Math 661 **Fall 2013**
Homework 5 **Drew Armstrong**

Problem 1 (Splitting Lemma). Let R be a commutative ring with 1 and consider a short exact sequence of R-modules:

$$0 \longrightarrow A \xrightarrow{q} B \xrightarrow{r} C \longrightarrow 0.$$

Prove that if there exists $t : B \to A$ such that $t \circ q$ is the identity on A, then $B \approx A \oplus C$. [Hint: Define a map $\varphi : B \to A \oplus C$ by $\varphi(b) := (t(b), r(b))$. To show that φ is injective, assume that $\varphi(b) = \varphi(b')$. Show that this implies $b - b' \in \ker r = \operatorname{im} q$, and hence $b - b' = q \circ t(b - b') = t(q(b) - q(b')) = t(0) = 0$. To show that φ is surjective consider $(a, c) \in A \oplus C$. Since r and t are surjective there exist $b, b' \in B$ such that $a = t(b)$ and $c = r(b')$. Now let $x = b' + q \circ t(b - b')$ and show that $\varphi(x) = (a, c)$.]

Problem 2. We say that a matrix $A \in GL(n, \mathbb{C})$ is unitary if $A^*A = I$, where A^* is the conjugate transpose. Let $U(n) \leq GL(n, \mathbb{C})$ denote the unitary group of unitary matrices.

(a) Prove that $U(n)$ is actually a group.
(b) Let $(x, y) = x^*y = \sum_i \overline{x_i} y_i$ be the standard Hermitian form on $\mathbb{C}^n$. Prove that $A \in GL(n, \mathbb{C})$ is unitary if and only if $(Ax, Ay) = (x, y)$ for all $x, y \in \mathbb{C}^n$.
(c) Prove that $A \in GL(n, K)$ is unitary if and only if its columns are orthonormal.
(d) Prove that every $A \in U(n)$ is conjugate in $U(n)$ to a diagonal matrix. [Hint: Let $A \in U(n)$. Since $\mathbb{C}$ is algebraically closed, A has an eigenvector, say $A\mathbf{v}_1 = \lambda \mathbf{v}_1$. Assume it is possible to extend this to an orthonormal basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ for $\mathbb{C}^n$ (which it is, via the Gram-Schmidt algorithm). Letting $P = \begin{pmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{pmatrix}$ gives us

$$P^{-1}AP = \left(\begin{array}{c|ccc} \lambda & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & A' & \\ 0 & & & \end{array}\right),$$

with $A' \in U(n-1)$. By induction, A' is conjugate in $U(n-1)$ to a diagonal matrix.]

Problem 3. Prove that the center of $GL(n, K)$ is the group of scalar matrices

$$Z(GL(n, K)) = \left\{\alpha I : \alpha \in K^\times\right\} \approx K^\times.$$

Prove that the center of $SL(n, K)$ is the group of n-th roots of unity

$$Z(SL(n, K)) = \{\alpha I : \alpha \in K, \alpha^n = 1\}.$$

Assuming that $\mathbb{F}_q^\times$ is a cyclic group (this is called the Primitive Root Theorem; please don't prove it), compute the order of $PSL(n, q)$.

Problem 4. Let $B \leq GL(n, K)$ be the Borel subgroup of upper triangular matrices, let $U \leq B$ be the subgroup of upper unitriangular matrices (i.e. with 1's on the diagonal) and let $T \leq B$ be the subgroup of diagonal matrices (called a maximal torus).

(a) Why is T called a torus?
(b) Prove that $B = T \ltimes U$.

(c) More generally, given $J = (n_1, \ldots, n_k) \in \mathbb{N}^k$ where $n_1 + n_2 + \cdots + n_k = n$ we define the parabolic subgroup

$$P_J = \begin{pmatrix} \boxed{*} & & & * \\ & \boxed{*} & & \\ & & \boxed{*} & \\ 0 & & & \boxed{*} \end{pmatrix} \leq GL(n, K)$$

where the diagonal blocks are square of sizes $n_1, n_2, \ldots, n_k$. We also define the unipotent radical and the Levi complement:

$$U_J = \begin{pmatrix} \boxed{I} & & & * \\ & \boxed{I} & & \\ & & \boxed{I} & \\ 0 & & & \boxed{I} \end{pmatrix} \leq P_J \quad \text{and} \quad L_J = \begin{pmatrix} \boxed{*} & & & 0 \\ & \boxed{*} & & \\ & & \boxed{*} & \\ 0 & & & \boxed{*} \end{pmatrix} \leq P_J.$$

Prove that $P_J = L_J \ltimes U_J$. [Hint: Consider the projection homomorphism $\varphi : P_J \to L_J$ Show that the kernel is U_J. Now consider any $g \in P_J$ and show that $g\varphi(g)^{-1} \in \ker \varphi = U_J$. It follows that $g \in U_J \cdot \varphi(g) \subseteq U_J L_J$.]