**Math 661** **Fall 2013**
**Homework 2 Solutions** **Drew Armstrong**

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**1.** Consider the lattice of subgroups $\mathscr{L}(G)$ of a group $G$. For each $H \in \mathscr{L}(G)$ and $g \in G$ let

$$gHg^{-1} := \{ghg^{-1} : h \in H\}.$$

(a) Show that $gHg^{-1}$ is a subgroup of $G$.
(b) Show that the map $G \times \mathscr{L}(G) \to \mathscr{L}(G)$ defined by $(g, H) \mapsto gHg^{-1}$ is a group action.
(c) The stabilizer of $H \in \mathscr{L}(G)$ under this action is called the normalizer of $H$:

$$N_G(H) := \{g \in G : gHg^{-1} = H\}.$$

Show that $N_G(H)$ is the largest subgroup of $G$ in which $H$ is normal.

*Proof.* For part (a) we will show that for all $a, b \in gHg^{-1}$ we have $ab^{-1} \in gHg^{-1}$. So suppose that $a, b \in gHg^{-1}$, say $a = gh_1g^{-1}$ and $b = gh_2g^{-1}$. Note that $b^{-1} = gh_2^{-1}g^{-1}$. Then we have

$$ab = (gh_1g^{-1})(gh_2g^{-1}) = g(h_1h_2^{-1})g^{-1} \in gHg^{-1},$$

as desired.

For part (b), let $(g, \bullet) : \mathscr{L}(G) \to \mathscr{L}(G)$ denote the map $H \mapsto gHg^{-1}$. We must show that for all $g, h \in G$ we have $(g, \bullet) \circ (h, \bullet) = (gh, \bullet)$. [In other words, the map $g \mapsto (g, \bullet)$ is a group homomorphism $G \to \text{Aut}(\mathscr{L}(G))$.] Indeed, for all $H \in \mathscr{L}(G)$ and for all $g, h \in G$ we have

$$(g, (h, H)) = (g, hHh^{-1}) = g(hHh^{-1})g^{-1} = (gh)H(gh)^{-1} = (gh, H).$$

For part (c), first note that $H$ is indeed a normal a subgroup of $N_G(H)$. Now suppose we have $H \lhd K \leq G$ for some $K$. We want to show that $K \leq N_G(H)$. Indeed, suppose $g \in K$. Then since $H \lhd K$ we have $gHg^{-1} = H$, which implies that $g \in N_G(H)$. □

**2.** Let $H \leq G$ be a subgroup.

(a) For each $a \in N_G(H)$, define a function $\theta_a : H \to G$ by $\theta_a(h) := aha^{-1}$. Show that $\theta_a$ is actually in $\text{Aut}(H)$, the group of automorphisms (i.e. self-isomorphisms) of $H$.
(b) Show that the map $\theta : N_G(H) \to \text{Aut}(H)$ is a group homomorphism.
(c) Show that the kernel of $\theta$ is the centralizer of $H$

$$C_G(H) := \{g \in G : ghg^{-1} = h \text{ for all } h \in H\}.$$

(d) Conclude that $C_G(H)$ is normal in $N_G(H)$ and that $N_G(H)/C_G(H)$ is isomorphic to a subgroup of Aut(H).

*Proof.* Let $a \in N_G(H)$. For part (a) we wish to show that the map $\theta_a : H \to G$ defined by $\theta_a(h) := aha^{-1}$ is actually in $\text{Aut}(H)$. Well, since $a \in N_G(a)$ we know that $aHa^{-1} = H$, hence for every $h \in H$ we have $aha^{-1} \in aHa^{-1} = H$. So the map $\theta_a$ sends $H$ to itself. The map is a homomorphism because for all $h_1, h_2 \in H$ we have

$$\theta_a(h_1)\theta_a(h_2) = (ah_1a^{-1})(ah_2a^{-1}) = a(h_1h_2)a^{-1} = \theta_a(h_1h_2).$$

Finally, the map is invertible because $\theta_a^{-1} = \theta_{a^{-1}}$. We conclude that $\theta_a \in \text{Aut}(H)$.

By part (a) we have a function $\theta : N_G(H) \to \text{Aut}(H)$ given by $a \mapsto \theta_a$. For part (b) we will show that $\theta$ is a homomorphism. Indeed, consider $a, b \in N_G(H)$. Then for all $h \in H$ we have

$$\theta_a \circ \theta_b(h) = \theta_a(bhb^{-1}) = a(bhb^{-1})a^{-1} = (ab)h(ab)^{-1} = \theta_{ab}(h),$$

which implies that $\theta_a \circ \theta_b = \theta_{ab}$ as functions. Indeed, note that $\theta_a : H \to H$ is the identity map if and only if $aha^{-1} = h$ for all $h \in H$, i.e., if and only if $a \in C_G(H)$.

For part (d) we apply the Fundamental Homomorphism Theorem to $\theta : N_G(H) \to \mathrm{Aut}(H)$ to conclude that

$$\frac{N_G(H)}{C_G(H)} = \frac{N_G(H)}{\ker\,\theta} \approx \mathrm{im}\,\theta \leq \mathrm{Aut}(H).$$

□

[Some Words (to ignore if you want): If $T \leq G$ is a maximal abelian subgroup of a compact Lie group $G$, then $N_G(T)/C_G(T)$ is called the Weyl group of $G$. It is important.]

**3.** Given two groups $H, K$ and a group homomorphism $\theta : H \to \mathrm{Aut}(K)$, we define the semidirect product of $H$ and $K$ with respect to $\theta$ as follows: The underlying set is the Cartesian product $H \times K$ and the group operation is

$$(h_1, k_1) \bullet (h_2, k_2) := (h_1 h_2, \theta_{h_2}^{-1}(k_1) k_2).$$

(a) Show that this is indeed a group. We call it $H \ltimes_\theta K$.
(b) Identify $H$ and $K$ with subgroups of $H \ltimes_\theta K$ via that maps $h \mapsto (h, 1_K)$ for $h \in H$ and $k \mapsto (1_H, k)$ for $k \in K$. Show that
$$H \cap K = 1, \quad K \trianglelefteq H \ltimes_\theta K, \quad \text{and} \quad HK = H \ltimes_\theta K.$$
(c) Furthermore, show that for all $h \in H$ and $k \in K$ we have $\theta_h(k) = hkh^{-1}$.

*Proof.* For part (a), we must show that the operation is associative, with an identity element and inverses. First note that $(1, 1)$ is an identity element because

$$(1,1) \bullet (h,k) = (1h, \theta_1^{-1}(1)k) = (1h, 1k) = (h,k).$$

Next observe that $(h,k)^{-1} = (h^{-1}, \theta_h(k^{-1}))$ because

$$\begin{aligned}(h,k) \bullet (h^{-1}, \theta_h(k^{-1})) &= (hh^{-1}, \theta_{h^{-1}}^{-1}(k)\theta_h(k^{-1})) \\ &= (1, \theta_h(k)\theta_h(k^{-1})) \\ &= (1, \theta_h(kk^{-1})) \\ &= (1, \theta_h(1)) \\ &= (1,1).\end{aligned}$$

Finally, observe that the operation is associative. Given $h_1, h_2, h_3 \in H$ and $k_1, k_2, k_3 \in K$ we have

$$\begin{aligned}[(h_1,k_1) \bullet (h_2,k_2)] \bullet (h_3,h_3) &= (h_1h_2, \theta_{h_2}^{-1}(k_1)k_2) \bullet (h_3, k_3) \\ &= ((h_1h_2)h_3, \theta_{h_3}^{-1}(\theta_{h_2}^{-1}(k_1)k_2)k_3) \\ &= ((h_1h_2)h_3, \theta_{h_3}^{-1} \circ \theta_{h_2}^{-1}(k_1)\theta_{h_3}^{-1}(k_2)k_3)\end{aligned}$$

and

$$\begin{aligned}(h_1,k_1) \bullet [(h_2,k_2) \bullet (h_3,k_3)] &= (h_1,k_1) \bullet (h_2h_3, \theta_{h_3}^{-1}(k_2)k_3) \\ &= (h_1(h_2h_3), \theta_{h_2h_3}^{-1}(k_1)\theta_{h_3}^{-1}(k_2)k_3).\end{aligned}$$

Since $(h_1h_2)h_3 = h_1(h_2h_3)$ and $\theta_{h_3}^{-1} \circ \theta_{h_2}^{-1} = (\theta_{h_2} \circ \theta_{h_3})^{-1} = \theta_{h_2h_3}^{-1}$, the two expressions are equal.

For part (b) we will identify $H$ and $K$ with a subgroups of $H \ltimes_\theta K$ via the maps $h \leftrightarrow (h, 1)$ and $k \leftrightarrow (1, k)$. Under these identifications we will show that the external semidirect product agrees with the corresponding internal semidirect product, i.e. $H \ltimes_\theta K = H \ltimes K$. There are three steps. First note that $H \ltimes_\theta K = HK$ because for all $h \in H$ and $k \in K$ we have

$$(h, k) = (h, 1) \bullet (1, k).$$

Next, note that $H \cap K = 1$ because the only element simultaneously of the form $(h, 1)$ and $(1, k)$ is the identity element $(1, 1)$. Finally, we will show that $K$ is normal in $H \ltimes_\theta K$. Indeed, for all $(1, a) \in K$ and $(h, k) \in H \ltimes_\theta K$ we have

$$\begin{aligned}
(h,k) \bullet (1,a) \bullet (h,k)^{-1} &= (h,k) \bullet (1,a) \bullet (h^{-1}, \theta_h(k^{-1})) \\
&= (h1, \theta_1^{-1}(k)a) \bullet (h^{-1}, \theta_h(k^{-1})) \\
&= (h, ka) \bullet (h^{-1}, \theta_h(k^{-1})) \\
&= (hh^{-1}, \theta_{h^{-1}}^{-1}(ka)\theta_h(k^{-1})) \\
&= (1, \theta_h(ka)\theta_h(k^{-1})) \\
&= (1, \theta_h(kak^{-1})) \in K.
\end{aligned}$$

For part (c), we will verify that conjugation action of $H$ on $K$ agrees with the homomorphism $\theta : H \to \mathrm{Aut}(K)$ that we used to externally define the semidirect product. Indeed, for all $h \in H$ and $k \in K$ we have

$$\begin{aligned}
\text{“}hkh^{-1}\text{”} &= (h,1) \bullet (1,k) \bullet (h,1)^{-1} \\
&= (h,1) \bullet (1,k) \bullet (h^{-1},1) \\
&= (h1, \theta_1^{-1}(1)k) \bullet (h^{-1},1) \\
&= (h,k) \bullet (h^{-1},1) \\
&= (hh^{-1}, \theta_{h^{-1}}^{-1}(k)1) \\
&= (1, \theta_h(k)) = \text{“}\theta_h(k)\text{”}.
\end{aligned}$$

□

[Here we took two groups $H, K$ that were not necessarily related and we created a group $G$ such that $H$ and $K$ embed in $G$ with the property that $G = H \ltimes K$. In order to do this, we needed a homomorphism $\theta : H \to \mathrm{Aut}(K)$. Without the homomorphism $\theta$ we could never get started. Semidirect products are the most basic way to create group extensions.]

**4.** Let $G$ be a group. If $G$ acts on a set $X$ via $\alpha : G \to \mathrm{Aut}(X)$, we say that the pair $(X, \alpha)$ is a $G$-set. Given two $G$-sets $(X, \alpha)$ and $(Y, \beta)$, we say that a function $\varphi : X \to Y$ is a $G$-set homomorphism if for all $g \in G$ the following diagram commutes:

$$\begin{array}{ccc}
X & \xrightarrow{\ \varphi\ } & Y \\
\Big\downarrow{\scriptstyle \alpha_g} & & \Big\downarrow{\scriptstyle \beta_g} \\
X & \xrightarrow[\ \varphi\ ]{} & Y
\end{array}$$

That is, for all $x \in X$ and $g \in G$ we have $\varphi(\beta_g(x)) = \alpha_g(\varphi(x))$. We say that two $G$-sets are isomorphic if there exists a bijective $G$-set homomorphism between them.

(a) If $\varphi : X \to Y$ is a $G$-set homomorphism, show that for all $x \in X$ we have

$$\mathrm{Stab}(x) \leq \mathrm{Stab}(\varphi(x)).$$

(b) If $\varphi : X \to Y$ is a $G$-set isomorphism, show that for all $x \in X$ we have

$$\text{Stab}(x) = \text{Stab}(\varphi(x)).$$

(c) Given a subgroup $H \leq G$ we put a $G$-set structure on $G/H$ by left-multiplication. Show that this $G$-set is **transitive**. Moreover, show that **any transitive** $G$**-set** is isomorphic to $G/H$ for some $H \leq G$.

*Proof.* For part (a), consider $g \in \text{Stab}(x)$. Then we have

$$\varphi(x) = \varphi(\alpha_g(x)) = \beta_g(\varphi(x)),$$

hence $g \in \text{Stab}(\varphi(x))$. For part (b), consider the inverse $G$-set homomorphism $\varphi^{-1} : Y \to X$. Applying part (a) to $\varphi(x) \in Y$ gives $\text{Stab}(\varphi(x)) \leq \text{Stab}(\varphi^{-1}(\varphi(x))) = \text{Stab}(x)$. Hence $\text{Stab}(x) = \text{Stab}(\varphi(x))$.

For part (c), consider $H \leq G$ and for each $g \in G$ define the map $\alpha_g : G/H \to G/H$ by $C \mapsto gC$. It is easy to check that $\alpha : G \to \text{Aut}(G/H)$ is a homomorphism. This action is transitive because for all $g_1H$ and $g_2H$ in $G/H$ we have $\alpha_{g_2g_1^{-1}}(g_1H) = g_2H$. Now let $X$ be any transitive $G$-set and let $H = \text{Stab}(x)$ for some $x \in X$. Recall that we have a bijection

$$\varphi : X \to G/H$$

defined by $\varphi(g(x)) := gH$. (We just replace the symbol $x$ by the symbol $H$.) Finally, observe that $\varphi$ is in fact a $G$-set isomorphism. Indeed, for all $g_1 \in G$ and $g_2(x) \in X$ we have

$$g_1(\varphi(g_2(x))) = g_1(g_2H) = (g_1g_2)H = \varphi((g_1g_2)(x)) = \varphi(g_1(g_2(x))).$$

□

**5.** Given a $G$-set $X$, let $\text{Aut}_G(X)$ denote the group of $G$-set automorphisms of $X$. In this problem you will show that for all **transitive** $G$-sets $X$ we have $\text{Aut}_G(X) \approx N_G(H)/H$, where $H$ is the stabilizer of a point and $N_G(H)$ is the normalizer of $H$ in $G$. By Problem 4(c) we can replace $X$ with $G/H$.

(a) Given $n \in N_G(H)$, show that **right multiplication** by $n^{-1}$ defines a $G$-set automorphism $G/H \to G/H$. Call this automorphism $\theta_n$.
(b) Show that $\theta : N_G(H) \to \text{Aut}_G(G/H)$ is a homomorphism with kernel $H$.
(c) Show that the homomorphism $\theta$ from part (b) is surjective. [Hint: Let $\varphi : G/H \to G/H$ be any $G$-set automorphism and suppose $\varphi(H) = n^{-1}H$. Use Problem 4(b) to conclude that $n \in N_G(H)$. Finally, show that for all $g \in G$ we have $\varphi(gH) = gHn^{-1}$.]
(d) If $G$ acts **freely** and **transitively** on $X$, conclude that $\text{Aut}_G(X) \approx G$.

*Proof.* For part (a), let $n \in N_G(H)$. First note that the rule $\theta_n(C) := Cn^{-1}$ actually defines a function $G/H \to G/H$. Indeed, given $gH \in G/H$ we have $gHn^{-1} = gn^{-1}H \in G/H$. Note that $\theta_n$ is a bijection because its has an inverse; namely, $\theta_n^{-1} = \theta_{n^{-1}}$. Finally, to see that $\theta_n$ is a $G$-set map, observe that for all $C \in G/H$ and for all $g \in G$ we have

$$g(\theta_n(C)) = g(Cn^{-1}) = (gC)n^{-1} = \theta_n(gC).$$

For part (b), observe that for all $C \in G/H$ and for all $m, n \in N_G(H)$ we have

$$\theta_{mn}(C) = C(mn)^{-1} = C(n^{-1}m^{-1}) = (Cn^{-1})m^{-1} = \theta_m \circ \theta_n(C),$$

hence $\theta$ is a homomorphism. Note that we have $\theta_n = \text{id}$ if and only if $gH = gHn^{-1}$ for all $g \in G$, which happens if and only if $n \in H$. Hence $\ker \theta = H$.

For part (c), let $\varphi : G/H \to G/H$ be any $G$-set isomorphism and suppose that $\varphi(H) = n^{-1}H$ for some $n \in G$. By Problem 4(b) we know that

$$H = \text{Stab}(H) = \text{Stab}(\varphi(H)) = \text{Stab}(n^{-1}H) = n^{-1}\text{Stab}(H)n = n^{-1}Hn,$$

hence $n \in N_G(H)$. Finally, for all $g \in G$ we have by assumption that $\varphi(gH) = g(\varphi(H))$, hence

$$\varphi(gH) = g(\varphi(H)) = g(n^{-1}H) = gHn^{-1}.$$

We conclude that the homomorphism $\theta : N_G(H) \to \text{Aut}_G(G/H)$ is surjective, and the Fundamental Homomorphism Theorem says that

$$\text{Aut}_G(G/H) = \text{im}\,\theta \approx \frac{N_G(H)}{\ker\theta} = \frac{N_G(H)}{H}.$$

For part (d), suppose that $G$ acts freely and transitively on $X$. In this case the stabilizer is trivial (i.e. $H = 1$), so we have

$$\text{Aut}_G(X) \approx \frac{N_G(1)}{1} = \frac{G}{1} \approx G.$$

□

[Thinking Problem: I originally made a mistake by thinking that $\text{Aut}_G(X)$ should be isomorphic to $\text{Aut}(G) \ltimes G$. Can anyone figure out what I meant to say? That is, if $G$ acts freely and transitively on a set $X$, is there some appropriate notion of "automorphism" such that "Aut"$(X) \approx \text{Aut}(G) \ltimes G$?]