Math 661 **Fall 2013**
Homework 2 **Drew Armstrong**

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1. Consider the lattice of subgroups $\mathscr{L}(G)$ of a group G. For each $H \in \mathscr{L}(G)$ and $g \in G$ let

$$gHg^{-1} := \{ghg^{-1} : h \in H\}.$$

(a) Show that gHg^{-1} is a subgroup of G.
(b) Show that the map $G \times \mathscr{L}(G) \to \mathscr{L}(G)$ defined by $(g, H) \mapsto gHg^{-1}$ is a group action.
(c) The stabilizer of $H \in \mathscr{L}(G)$ under this action is called the normalizer of H:

$$N_G(H) := \{g \in G : gHg^{-1} = H\}.$$

Show that $N_G(H)$ is the largest subgroup of G in which H is normal.

2. Let $H \leq G$ be a subgroup.

(a) For each $a \in N_G(H)$, define a function $\theta_a : H \to G$ by $\theta_a(h) := aha^{-1}$. Show that θ_a is actually in $\mathrm{Aut}(H)$, the group of automorphisms (i.e. self-isomorphisms) of H.
(b) Show that the map $\theta : N_G(H) \to \mathrm{Aut}(H)$ is a group homomorphism.
(c) Show that the kernel of θ is the centralizer of H

$$C_G(H) := \{g \in G : ghg^{-1} = h \text{ for all } h \in H\}.$$

(d) Conclude that $C_G(H)$ is normal in $N_G(H)$ and that $N_G(H)/C_G(H)$ is isomorphic to a subgroup of Aut(H).

[Some Words (to ignore if you want): If $T \leq G$ is a maximal abelian subgroup of a compact Lie group G, then $N_G(T)/C_G(T)$ is called the Weyl group of G. It is important.]

3. Given two groups H, K and a group homomorphism $\theta : H \to \mathrm{Aut}(K)$, we define the semidirect product of H and K with respect to θ as follows: The underlying set is the Cartesian product $H \times K$ and the group operation is

$$(h_1, k_1) \bullet (h_2, k_2) := (h_1 h_2, \theta_{h_2}^{-1}(k_1) k_2).$$

(a) Show that this is indeed a group. We call it $H \ltimes_\theta K$.
(b) Identify H and K with subgroups of $H \ltimes_\theta K$ via that maps $h \mapsto (h, 1_K)$ for $h \in H$ and $k \mapsto (1_H, k)$ for $k \in K$. Show that

$$H \cap K = 1, \quad K \trianglelefteq H \ltimes_\theta K, \quad \text{and} \quad HK = H \ltimes_\theta K.$$

(c) Furthermore, show that for all $h \in H$ and $k \in K$ we have $\theta_h(k) = hkh^{-1}$.

4. Let G be a group. If G acts on a set X via $\alpha : G \to \mathrm{Aut}(X)$, we say that the pair (X, α) is a G-set. Given two G-sets (X, α) and (Y, β), we say that a function $\varphi : X \to Y$ is a G-set homomorphism if for all $g \in G$ the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\alpha_g"'] & Y \arrow[d, "\beta_g"] \\ X \arrow[r, "\varphi"'] & Y \end{tikzcd}$$

That is, for all $x \in X$ and $g \in G$ we have $\varphi(\beta_g(x)) = \alpha_g(\varphi(x))$. We say that two G-sets are isomorphic if there exists a bijective G-set homomorphism between them.

(a) If $\varphi : X \to Y$ is a G-set homomorphism, show that for all $x \in X$ we have

$$\mathrm{Stab}(x) \leq \mathrm{Stab}(\varphi(x)).$$

(b) If $\varphi : X \to Y$ is a G-set isomorphism, show that for all $x \in X$ we have

$$\mathrm{Stab}(x) = \mathrm{Stab}(\varphi(x)).$$

(c) Given a subgroup $H \leq G$ we put a G-set structure on G/H by left-multiplication. Show that this G-set is **transitive**. Moreover, show that **any transitive G-set** is isomorphic to G/H for some $H \leq G$.

5. Given a G-set X, let $\mathrm{Aut}_G(X)$ denote the group of G-set automorphisms of X. In this problem you will show that for all **transitive** G-sets X we have $\mathrm{Aut}_G(X) \approx N_G(H)/H$, where H is the stabilizer of a point and $N_G(H)$ is the normalizer of H in G. By Problem 4(c) we can replace X with G/H.

(a) Given $n \in N_G(H)$, show that **right multiplication** by n^{-1} defines a G-set automorphism $G/H \to G/H$. Call this automorphism θ_n.

(b) Show that $\theta : N_G(H) \to \mathrm{Aut}_G(G/H)$ is a homomorphism with kernel H.

(c) Show that the homomorphism θ from part (b) is surjective. [Hint: Let $\varphi : G/H \to G/H$ be any G-set automorphism and suppose $\varphi(H) = n^{-1}H$. Use Problem 4(b) to conclude that $n \in N_G(H)$. Finally, show that for all $g \in G$ we have $\varphi(gH) = gHn^{-1}$.]

(d) If G acts **freely** and **transitively** on X, conclude that $\mathrm{Aut}_G(X) \approx G$.