Math 661 **Fall 2013**
Homework 1 (Review of 561/562) **Drew Armstrong**

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1. Let $H \leq G$ be a subgroup. Call the identity element 1.

(a) State the definition of equivalence relation.
(b) Define a relation on G by setting $a \sim_H b \Leftrightarrow a^{-1}b \in H$. Prove that this is an **equivalence** relation on G.
(c) Given an element $g \in G$ we define the left coset $gH := \{gh : h \in H\}$. Prove that $a \sim_H b$ **if and only if** $aH = bH$.
(d) Prove that the map $g \mapsto ag$ is a **bijection** from H to aH.
(e) If $|G|$ is finite, prove that $|H|$ **divides** $|G|$.
(f) For all $a \in G$ prove that $a^{|G|} = 1$. [Hint: Use part (e).]
(g) Finally, let $G = (\mathbb{Z}/n\mathbb{Z})^\times$ (i.e. the group of units of the ring $\mathbb{Z}/n\mathbb{Z}$). What does the result of (f) say in this case?

2. Let $K \leq G$ be a subgroup and let G/K denote the **set** of left cosets of K. Consider the surjective **map of sets** $\varphi : G \to G/K$ defined by $\varphi(a) := aK$.

(a) **Suppose** there exists some group operation on G/K such that φ is a group homomorphism. In this case, what is the identity element of G/K? What is $\ker \varphi$?
(b) If G' is any group and $\psi : G \to G'$ is any group homomorphism, prove that $\ker \psi$ is a **normal** subgroup of G (i.e. prove that $gkg^{-1} \in \ker \psi$ for all $k \in \ker \psi$).
(c) Now suppose that $K \trianglelefteq G$ is normal (i.e. suppose that $gkg^{-1} \in K$ for all $k \in K$). In this case, prove that the operation $(G/K) \times (G/K) \to G/K$ given by $(aK, bK) \mapsto (ab)K$ is **well-defined**.
(d) Moreover, prove that this operation makes G/K into a **group**. (And hence the original φ is a group homomorphism.)
(e) Finally, let $H \leq G$ be any subgroup. Prove that H is **normal if and only if** there exists a group G' and a group homomorphism $\mu : G \to G'$ such that $\ker \mu = H$.

3. Let R be a commutative ring with 1 and let $I \leq R$ be an ideal.

(a) Finish the sentence: We say that R is an integral domain if ...
(b) Finish the sentence: We say that I is a prime ideal if ...
(c) If R/I is an integral domain, prove that I must be prime.
(d) If I is prime, prove that R/I must be an integral domain.
(e) Finish the sentence: We say that R is a field if ...
(f) Finish the sentence: We say that I is a maximal ideal if ...
(g) If R/I is a field, prove that I must be maximal.
(h) If I is maximal, prove that R/I must be a field.
(i) Finally, explain why every maximal ideal is prime.

4. Let $F \subseteq K$ be a field extension with $\alpha \in K$, and consider the ring of polynomials $F[x]$. Let $\varphi_\alpha : F[x] \to K$ be the ring homomorphism defined by $\varphi_\alpha(x) := \alpha$ and $\varphi_\alpha(a) := a$ for all $a \in F$. We use the notation $\varphi_\alpha(f(x)) = f(\alpha)$.

(a) Prove that $I := \ker \varphi_\alpha$ is an ideal of $F[x]$.

(b) Prove that this ideal $I \leq F[x]$ is **principal**. [Hint: If $I \neq (0)$ then choose $0 \neq f(x) \in I$ with minimal degree. Show that $I \subseteq (f(x))$.]

(c) By part (b) we can write $I = (m_\alpha(x))$ for some monic $m_\alpha(x) \in F[x]$. Prove that this $m_\alpha(x)$ is **irreducible** over F.

(d) Use the first isomorphism theorem to prove that $F \subseteq \operatorname{im} \varphi_\alpha \subseteq K$ is a **field**.

(e) If L is any intermediate field $F \subseteq L \subseteq K$ such that $\alpha \in L$, prove that $\operatorname{im} \varphi_\alpha \subseteq L$. (Hence $\operatorname{im} \varphi_\alpha$ is the **smallest** subfield of K containing F and α.)