2/10/15

HW 1 due now.
Quiz 2 in class Thursday (~20 min)
based on course notes from Jan 22
to Feb 5 ("Trigonometry").

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We have now seen how sine waves explain the phenomenon of beats, and how beats explain Pythagoras' Observation (Plomp-Levelt curves).

We still have not answered the question:

Why Sine Waves?

In other words, why do sine waves accurately model the phenomena of sound?

The answer comes from Physics via the study of differential equations, the most important for us being:

The Wave Equation.

This is our next topic.

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We begin with the most basic differential equation. Let $f(t)$ be a quantity whose rate of growth is directly proportional to its current value:

$$f'(t) = a \cdot f(t)$$

for some constant $a$.

How can we solve this? Note that such a function must be infinitely differentiable because

$$\begin{aligned} f''(t) &= \left(f'(t)\right)' \\ &= \left(a f(t)\right)' \\ &= a f'(t) \end{aligned}$$

etc.

Thus we can assume [omitting technical details] that $f(t)$ has a power series expansion

$$f(t) = \sum_{n \geq 0} c_n t^n = c_0 + c_1 t + c_2 t^2 + \cdots .$$

To solve for $f(t)$ means to solve for the coefficients $c_0, c_1, c_2, \ldots$,

Note that

$$af(t) = ac_0 + ac_1 t + ac_2 t^2 + \cdots$$

$$f'(t) = c_1 + 2c_2 t + 3c_3 t^2 + \cdots$$

and hence

$$c_1 = ac_0$$
$$2c_2 = ac_1 = a^2 c_0 \implies c_2 = a^2 c_0 / 2$$
$$3c_3 = ac_2 = a^3 c_0 / 2 \implies c_3 = a^3 c_0 / 6$$

and in general we have

$$c_n = a^n c_0 / (2 \cdot 3 \cdots n) = \frac{a^n c_0}{n!}$$

We conclude that the solution is

$$f(t) = c_0 + ac_0 t + a^2 c_0 \frac{t^2}{2} + a^3 c_0 \frac{t^3}{6} + \cdots$$
$$= c_0 \left[ 1 + (at) + \frac{(at)^2}{2} + \frac{(at)^3}{6} + \cdots \right]$$

$$= c_0 \sum_{n \geq 0} \frac{(at)^n}{n!}$$

Great! We solved the problem. Do you recognize this solution?

Recall the following fact/definition:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!} + \cdots .$$

Thus our solution is

$$f(t) = c_0 e^{at}.$$

What is $c_0$? It's just the initial value of $f$:

$$f(0) = c_0 e^0 = c_0.$$

Finally, we have

$$f(t) = f(0) e^{at}.$$

///

More generally, assume that $\bar{F}(t)$ is a <u>parametrized</u> <u>curve</u> in $\mathbb{R}^2$:

$$\bar{F}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}$$

where $f_1$ and $f_2$ are functions $\mathbb{R} \to \mathbb{R}$.

We can draw such a curve

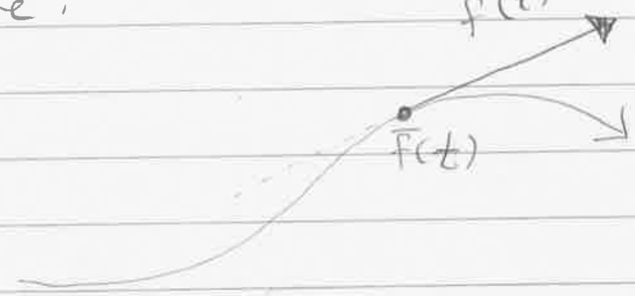

If we define the derivative by

$$\bar{F}'(t) := \begin{pmatrix} f_1'(t) \\ f_2'(t) \end{pmatrix},$$

what does this mean geo<u>metrically</u>?

A: · $\bar{F}'(t)$ is the instantaneous velocity vector at $\bar{F}(t)$.

Picture:

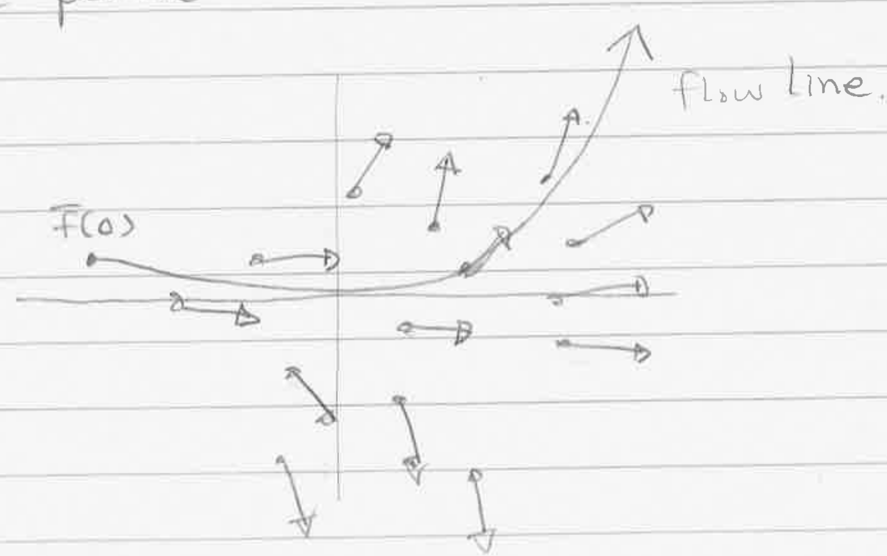



It is always tangent to the curve. The instaneous <u>speed</u> is the length of the velocity vector

$$\| \bar{F}'(t) \| = \sqrt{f_1(t)^2 + f_2(t)^2} \,.$$

[ If the speed is always 1 we say the curve is "parametrized by arc length". ]

Now suppose we have a <u>vector</u> <u>field</u> in the plane



At each point in $\mathbb{R}^2$ we attach a vector in $\mathbb{R}^2$, so a vector field is a function

$$\underset{\phantom{x}}{\Phi} : \underset{\text{points}}{\mathbb{R}^2} \longrightarrow \underset{\text{vectors}}{\mathbb{R}^2}$$

Q: If we drop a cork in the vector field and let it flow, what path will it take?

A: Let $\bar{f}(t)$ be the path of the cork. At each moment, the velocity of the cork agrees with the vector field, i.e.,

$$\bar{f}'(t) = \Phi(\bar{f}(t)).$$

To integrate the vector field means to solve this system of differential equations.

This can be quite hard, so we will restrict our attention to linear vector fields:

$$A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2.$$

Now we can think of $A$ as a matrix.

Problem: Given a $2 \times 2$ matrix $A$ representing a linear vector field in the plane, compute the flow. That is, given an initial point $\bar{c}_0 \in \mathbb{R}^2$ find the curve $\bar{f}(t)$ such that

- $\bar{f}(0) = \bar{c}_0$
- $\bar{f}'(t) = A \cdot \bar{f}(t)$.

☆ Amazing Fact: The solution to this system of 2 differential equations is "the same" as the previous problem

[ Recall: The solution to

- $f(0) = c_0$
- $f'(t) = a f(t)$

was $f(t) = c_0 e^{at}$. ]

The solution is

$$\bar{f}(t) = e^{At} \cdot \bar{c}_0 .$$

//

What?! We need to discuss this.

In general, if $X$ is a square matrix we can define its exponential

$$e^X := I + X + \frac{1}{2}X^2 + \frac{1}{6}X^3 + \cdots + \frac{1}{n!}X^n + \cdots$$

One can show [technical details omitted] that this series always converges.

In our case, $e^{At}$ is a $2\times 2$ matrix. Since the initial condition $\overline{c}_0$ is a $2\times 1$ column vector we must multiply $e^{At}$ on the left:

$$\overline{f}(t) = e^{At}\cdot\overline{c}_0$$

$$\begin{pmatrix} \cdot \\ \cdot \end{pmatrix} = \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}\begin{pmatrix} \cdot \\ \cdot \end{pmatrix}$$

It is easy to show that this solution works by differentiating the power series.

2/12/15

HW 2 : TBA.
Today: Quiz 2 ( 20 minutes).

Last time we discussed vector fields

$$\Phi : \mathbb{R}^2 \to \mathbb{R}^2.$$

If the field is linear (i.e. $\Phi(\alpha\bar{x} + \beta\bar{y}) = \alpha\Phi(\bar{x}) + \beta\Phi(\bar{y})\ \forall\ \bar{x}, \bar{y} \in \mathbb{R}^2,\ \alpha, \beta \in \mathbb{R}$) then it can be expressed as a matrix

$$A : \mathbb{R}^2 \to \mathbb{R}^2$$

We are interested in "integrating" the field to compute its "flow". That is, given a point $\bar{c}_0 \in \mathbb{R}^2$ we want to find a parametrized curve $\bar{f}(t)$ such that

- $\bar{f}(0) = \bar{c}_0$
- $\bar{f}'(t) = A \cdot \bar{f}(t)$

We saw last time that the solution has a nice closed form:

$$\bar{f}(t) = e^{At} \cdot \bar{c}_0 \,,$$

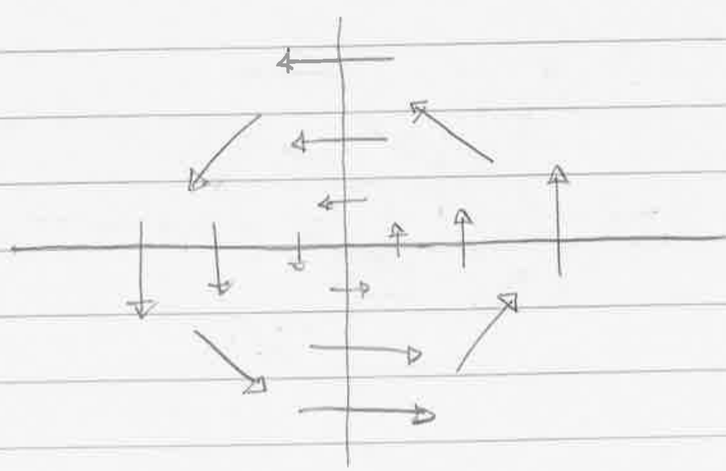
where $e^{At}$ is the $2 \times 2$ matrix defined by the power series

$$e^{At} := I + At + A^2 \frac{t^2}{2} + A^3 \frac{t^3}{6} + \cdots + A^n \frac{t^n}{n!} + \cdots$$

[Remark: This works for linear vector fields in any number of dimensions; not just 2. You already know that it works in 1 dimension! ]

Example: Consider the vector field $\mathbb{R}^2 \to \mathbb{R}^2$ defined by $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Picture: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$



It seems to go around in circles. For the initial condition $\bar{x}_0 = (x_0, y_0)$, the flow $\bar{x}(t) = (x(t), y(t))$ is given by

$$\bar{x}(t) = e^{At} \cdot \bar{x}_0$$

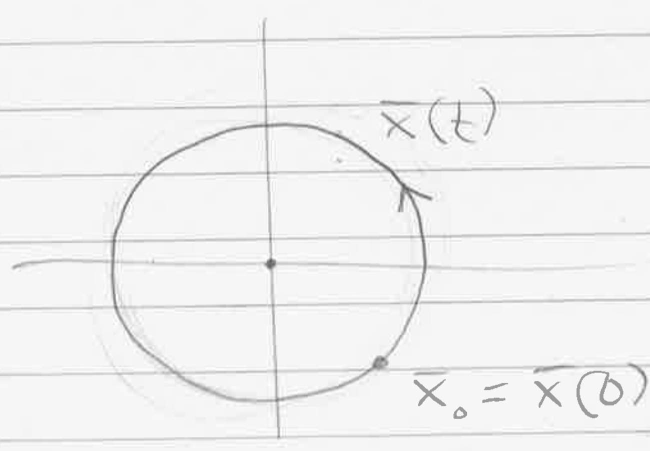
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} t} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$= \begin{pmatrix} x_0 \cos t - y_0 \sin t \\ x_0 \sin t + y_0 \cos t \end{pmatrix}$$

Note that $\bar{x}(t)$ moves in a circle traveling counterclockwise from $\bar{x}_0$.

Picture:



What is the <u>speed</u> of the flow?

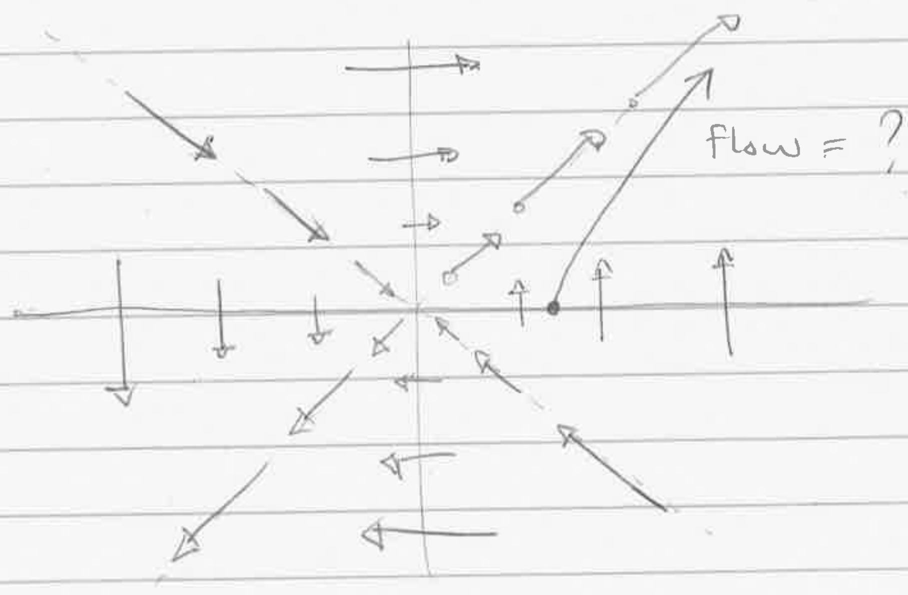
Exercise: Show that the speed

$$\|\bar{x}'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2}$$

is constant and equal to the radius of the circle: $\sqrt{x_0^2+y_0^2}$.

Another Example: $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Sketch: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$



Let $\bar{x}(t) = (x(t), y(t))$ be the flow with initial condition $\bar{x}(0) = (x_0, y_0)$. Then the solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{\begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}} \cdot \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

To solve this we must compute $e^{\begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}}$.

First note that $A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; $A^3 = A$, and in general we have

$$A^n = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & n \text{ even} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & n \text{ odd} \end{cases}$$

So $e^{At}$ is given by

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} t + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{t^2}{2} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{t^3}{6} + \cdots$$

$$= \begin{pmatrix} 1 + \frac{t^2}{2} + \frac{t^4}{24} + \cdots & t + \frac{t^3}{6} + \frac{t^5}{120} + \cdots \\ t + \frac{t^3}{6} + \frac{t^5}{120} + \cdots & 1 + \frac{t^2}{2} + \frac{t^4}{24} + \cdots \end{pmatrix}$$

Do you recognize these power series? Let's give them names:

$$\cosh(t) := 1 + \frac{t^2}{2} + \frac{t^4}{24} + \cdots + \frac{t^{2n}}{(2n)!} + \cdots$$

$$\sinh(t) := t + \frac{t^3}{6} + \frac{t^5}{120} + \cdots + \frac{t^{2n-1}}{(2n-1)!} + \cdots$$

They can be expressed easily in terms of

$$e^t = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \cdots$$

$$e^{-t} = 1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \cdots$$

Note that

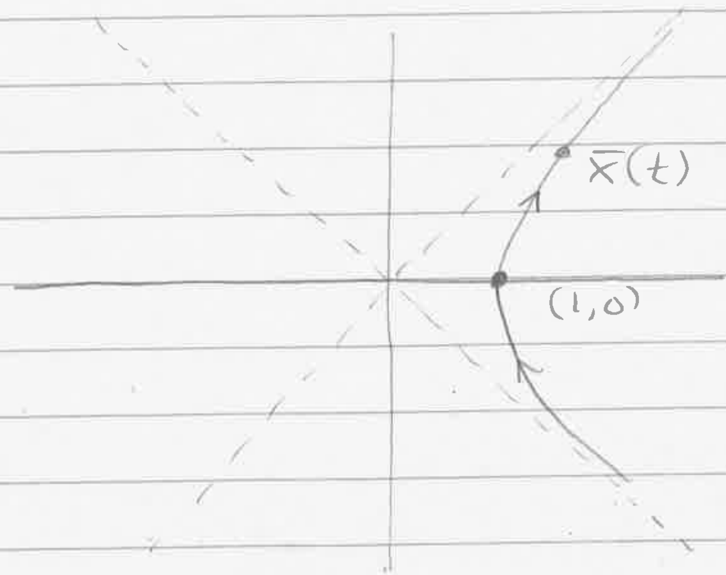
$$\cosh(t) = \frac{e^t + e^{-t}}{2} \quad \& \quad \sinh(t) = \frac{e^t - e^{-t}}{2}.$$

If our initial condition is $\bar{r}(0) = (1, 0)$ then the flow is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix}.$$

What does this curve look like?




It's a brach of a hyperbola! This explains the "h" in cosh & sinh.

Q: What explains the cos & sin in cosh & sinh?

Recall Euler's Formula:

$$e^{it} = \cos t + i \sin t$$

$$e^{-it} = \cos t - i \sin t.$$

Adding and subtracting these two equations gives us

$$\cos t = \frac{e^{it}+e^{-it}}{2} \quad \& \quad \sin t = \frac{e^{it}-e^{-it}}{2i}$$

In other words, we have

$$\cos(t) = \cosh(it) \quad \& \quad i\sin(t) = \sinh(it).$$

That explains the "cos" & "sin".

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Thinking Problem:

How can we tell from the matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \& \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

that one goes around in circles and one goes away to $\infty$?

2/17/15

HW 2 : TBA.

Recall that the complete solution to a linear system of 1st order ode's

$$\bar{x}'(t) = A \cdot \bar{x}(t)$$

is given by

$$\bar{x}(t) = e^{At} \cdot \bar{x}(0)$$

In general it is difficult to compute the matrix exponential

$$e^{At} := \sum_{n \geq 0} A^n \frac{t^n}{n!},$$

so in practice we use a different technique.

Definition: Let $A$ be a square $n \times n$ matrix. We say that $\bar{x} \in \mathbb{R}^n$ is an <u>eigenvector</u> for $A$ if

$$\boxed{A \cdot \bar{x} = \lambda \bar{x}}$$

for some constant $\lambda \in \mathbb{R}$.

This $\lambda$ is called the eigenvalue corresponding to the eigenvector $\bar{x}$.

Exercise: If $\bar{x}$ is an eigenvector of $A$, show that $\bar{x}$ is also an eigenvector of $2A^2 + A - I$.

Solution: Suppose $A\bar{x} = \lambda\bar{x}$. Then

$$(2A^2 + A - I)\bar{x}$$

$$= 2A^2 \cdot \bar{x} + A\bar{x} - I\bar{x}$$

$$= 2A(A\bar{x}) + \lambda \cdot \bar{x} - 1\bar{x}$$

$$= 2A(\lambda\bar{x}) + \lambda\bar{x} - 1\bar{x}$$

$$= 2\lambda A\bar{x} + \lambda\bar{x} - 1\bar{x}$$

$$= 2\lambda\lambda\bar{x} + \lambda\bar{x} - 1\bar{x}$$

$$= (2\lambda^2 + \lambda - 1)\bar{x}.$$

Yes. It is an eigenvector with eigenvalue $2\lambda^2 + \lambda - 1$. ///

In general, if $A\bar{x} = \lambda \bar{x}$ and $f(x)$ is any polynomial then $\bar{x}$ is also an eigenvector of the matrix $f(A)$. In fact,

$$f(A)\cdot \bar{x} = f(\lambda)\cdot \bar{x}$$

So What?

The same works for convergent power series.

☆ Useful Fact:

If $A\bar{x} = \lambda\bar{x}$ then we have

$$e^{At}\cdot \bar{x} = e^{\lambda t}\cdot \bar{x}.$$

<u>Proof:</u>

$$e^{At}\cdot\bar{x} = \left(\sum A^n \frac{t^n}{n!}\right)\cdot \bar{x}$$

$$= \left(\sum \frac{t^n}{n!}(A^n\bar{x})\right)$$

$$= \left(\sum \frac{t^n}{n!}(\lambda^n\bar{x})\right)$$

$$= \left(\sum \frac{(t\lambda)^n}{n!}\right)\bar{x} = e^{\lambda t}\cdot\bar{x}$$

///

This leads to a method for solving linear systems of ode's:

☆ "To solve $\bar{x}'(t) = A \cdot \bar{x}(t)$, express the initial condition $\bar{x}(0)$ in terms of eigenvectors of $A$".

Example: Solve the system

- $\bar{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
- $\bar{x}'(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \bar{x}(t)$

The eigenvectors of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -1 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Express $\bar{x}(0) = (1, 0)$ in terms of eigenvectors:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Therefore, the solution is

$$\bar{x}(t) = e^{At} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= e^{At}\left[\frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right]$$

$$= \frac{1}{2} e^{At}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2} e^{At}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \frac{1}{2} e^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2} e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} (e^{t} + e^{-t})/2 \\ (e^{t} - e^{-t})/2 \end{pmatrix}$$

$$= \begin{pmatrix} \cosh(t) \\ \sinh(t) \end{pmatrix}$$

///

Example: Solve the system

- $\bar{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
- $\bar{x}'(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \bar{x}(t)$.

The eigenvectors of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ are:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} i \\ 1 \end{pmatrix} = i \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} -i \\ 1 \end{pmatrix} = -i \begin{pmatrix} 1 \\ i \end{pmatrix}$$

Express $\bar{x}(0) = (1, 0)$ in terms of the eigenvectors:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ -i \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

Therefore, the solution is

$$\bar{x}(t) = e^{At} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= e^{At}\left[ \frac{1}{2}\begin{pmatrix} 1 \\ -i \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ i \end{pmatrix} \right]$$

$$= \frac{1}{2} e^{At}\begin{pmatrix} 1 \\ -i \end{pmatrix} + \frac{1}{2} e^{At}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$= \frac{1}{2} e^{it}\begin{pmatrix} 1 \\ -i \end{pmatrix} + \frac{1}{2} e^{-it}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$= \begin{pmatrix} (e^{it} + e^{-it})/2 \\ (e^{it} - e^{-it})/2i \end{pmatrix}$$

$$= \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \qquad ///$$

This is a powerful method. We will use it again later to solve the wave equation.

Back to music.

Recall that a "wave" is any disturbance that propagates through an elastic medium, and recall that elastics are described by

☆ Hooke's Law:

Let $x(t)$ be the position of a particle subject to an elastic force with equilibrium at $x=0$. Then we have

$$\text{force} = -k \cdot x(t)$$

where $k$ is the "spring constant". If the particle has mass $m$, then Newton's 2nd Law says

$$m \cdot x''(t) = -k \cdot x(t)$$

$$x''(t) = -\frac{k}{m} \cdot x(t)$$

Can we solve this?

To simplify notation, let $\omega^2 = k/m$ :

$$x''(t) = -\omega^2 \cdot x(t)$$

We don't know how to solve 2nd order ode's so we use a trick to replace this with a system of two 1st order ode's.

TRICK: Let $\begin{cases} u_1(t) := \omega x(t) \\ u_2(t) := -x'(t) \end{cases}$.

Then we have

$$u_1'(t) = \omega \cdot x'(t) = -\omega \cdot u_2(t)$$

$$u_2'(t) = -x''(t) = \omega^2 \cdot x(t) = \omega \cdot u_1(t).$$

This can be written as a linear system

$$\begin{pmatrix} u_1'(t) \\ u_2'(t) \end{pmatrix} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$$

whose solution we already know.

Given initial position $x(0)$ and velocity $x'(0)$, the solution is

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = e^{\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} t} \cdot \begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix}$$

$$\begin{pmatrix} \omega x(t) \\ -x'(t) \end{pmatrix} = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix} \begin{pmatrix} \omega x(0) \\ -x'(0) \end{pmatrix}$$

This gives us two equations

$$\omega x(t) = \omega x(0) \cos(\omega t) + x'(0) \sin(\omega t)$$

$$-x'(t) = \omega x(0) \sin(\omega t) - x'(0) \cos(\omega t).$$

The second is just a consequence of the first so we throw it away to get

$$\boxed{x(t) = x(0) \cos(\omega t) + \frac{x'(0)}{\omega} \sin(\omega t)}$$

This is why we use sine waves to model sound.

Discussion:

- If the spring constant $k$ is positive, then the particle will oscillate with "angular velocity" $\omega = \sqrt{k/m}$, measured in cycles per second.

- What happens if $k < 0$?

- Do we know how to solve the general 2nd order equation

$$m x''(t) + \mu x'(t) + k x(t) = 0 \quad ?$$

What is the physical meaning of $\mu$?

2/19/15

HW 2 : TBA on Tuesday.

Current Goal : The Wave Equation.

Recall : If a particle of mass $m$ is subject to an elastic force of stiffness $k$, then Hooke's Law says

$$m x''(t) = -k x(t).$$

where $x(t)$ is the position of the particle at time $t$. For convenience we set $\omega^2 := -k/m$ so that

$$x''(t) = -\omega^2 x(t),$$

Then we do a little trick : Define

$$u_1(t) := \omega x(t)$$
$$u_2(t) := -x'(t)$$

and observe that

$$u_1'(t) = -\omega u_2(t)$$
$$u_2'(t) = \omega u_1(t) .$$

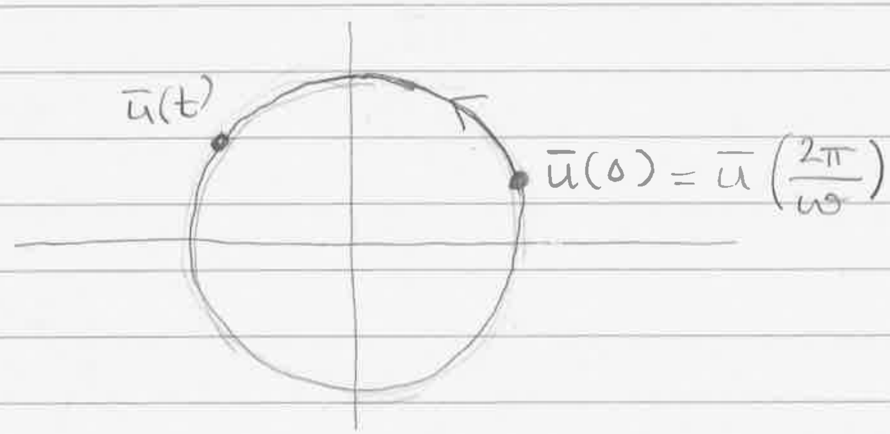
We can write this as a linear system

$$\begin{pmatrix} u_1'(t) \\ u_2'(t) \end{pmatrix} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$$

whose solution we know:

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix} \begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix}$$

The picture of this flow is called the "phase portrait" of the system:




Note that the length $\|\bar{u}(t)\|$ is preserved by the flow; i.e., for all times $t$ we have

$$\|\bar{u}(t)\|^2 = \|\bar{u}(0)\|^2 = u_1(0)^2 + u_2(0)^2$$
$$= \omega^2 x(0)^2 + x'(0)^2.$$

What is the physical significance of this quantity?

Does it *have* some physical significance?

You might recognize it if I multiply by the constant $1/2m$:

$$\frac{1}{2m}\|\bar{u}(t)\|^2 = \frac{1}{2} k\, x(t)^2 + \frac{1}{2} m\, x'(t)^2.$$

This mysterious conserved quantity is called *energy*. It is slightly imaginary but still useful.

*Example*: If the particle begins at rest (i.e., $x'(0) = 0$) then its total energy is $1/2 \cdot k\, x(0)^2$. [This is called "potential energy".]

When it reaches $x(t) = 0$, the total energy will be $1/2 \cdot m\, x'(t)^2$. [This is called "kinetic energy".]

Since energy is conserved we have

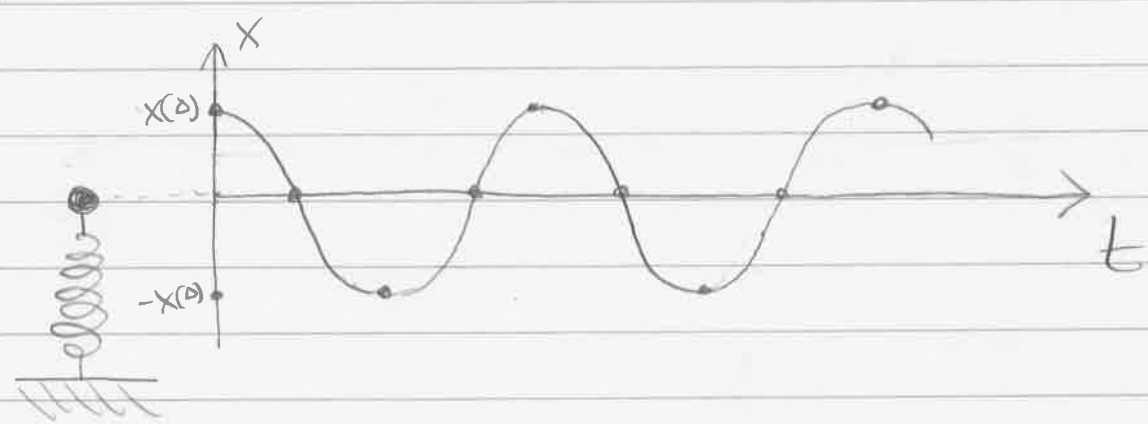
$$\frac{1}{2} m x'(t)^2 = \frac{1}{2} k x(0)^2$$

$$x'(t)^2 = \frac{k}{m} x(0)^2.$$

$$x'(t) = \pm \omega x(0)$$

This is the maximum speed attained by the particle.

Picture:




What is the equation of the motion?

$$u_1(t) = u_1(0)\cos(\omega t) - u_2(0)\sin(\omega t)$$

$$\omega x(t) = \omega x(0)\cos(\omega t) + x'(0)\sin(\omega t).$$

$$x(t) = x(0)\cos(\omega t) + \frac{x'(0)}{\omega}\sin(\omega t).$$

If $x'(0) = 0$ we get

$$x(t) = x(0)\cos(\omega t),$$

as the picture suggests.

Q: What if $x'(0) \neq 0$ ?

A: I claim that $x(t)$ is still a pure sine wave, just "out of phase".


In other words, we want to express

$$x(t) = x(0)\cos(\omega t) + \frac{x'(0)}{\omega}\sin(\omega t)$$

in the form $x(t) = a \cdot \cos(\omega t + \varphi)$, for some constants $a$ and $\varphi$.

Is this even possible?

Assume that $x(t) = a \cdot \cos(\omega t + \varphi)$ and try to solve for $a, \varphi$. Using the angle sum formula gives

$$x(t) = a \cdot \cos\varphi \cos(\omega t) - a \sin\varphi \sin(\omega t).$$

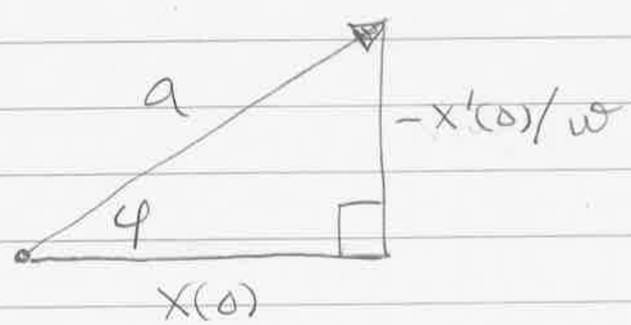
On the other hand, we know that

$$x(t) = x(0)\cos(\omega t) + \frac{x'(0)}{\omega}\sin(\omega t).$$

Thus we want to solve

$$\begin{pmatrix} x(0) \\ -x'(0)/\omega \end{pmatrix} = a \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}.$$

And this is easy to do:



i.e., $\begin{cases} a^2 = x(0)^2 + \frac{1}{\omega^2} x'(0)^2 \\ \varphi = \arctan\left(\frac{-x'(0)}{\omega x(0)}\right) \end{cases}$

In summary, the solution of

$$x''(t) + \omega^2 x(t) = 0$$

is given by

$$x(t) = \sqrt{x(0)^2 + \frac{1}{\omega^2} x'(0)^2} \cdot \cos\left(\omega t + \tan^{-1}\left(\frac{-x'(0)}{\omega x(0)}\right)\right).$$

This system is called the

"simple harmonic oscillator".

---

Before moving on, let's investigate the general 2nd order o.d.e. :

$$m x''(t) + \mu x'(t) + k x(t) = 0$$

called the "damped harmonic oscillator".
[The constant $\mu$ represents friction.]

I recommend we define $\gamma := \mu/m$, $\omega := k/m$ so that

$$x''(t) + \gamma x'(t) + \omega^2 x(t) = 0.$$

We'll use the same trick as before:

$$u_1(t) := \omega x(t)$$
$$u_2(t) := -x'(t)$$

But this time we have

$$\begin{pmatrix} u_1'(t) \\ u_2'(t) \end{pmatrix} = \begin{pmatrix} 0 & -\omega \\ \omega & -\gamma \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$$

The behavior of the system is governed by the eigenvalues & eigenvectors of

$$A := \begin{pmatrix} 0 & -\omega \\ \omega & -\gamma \end{pmatrix}$$

How about I just give you the evalues & e.vectors?

Here they are: Let $\lambda_1, \lambda_2$ be the two roots of the equation

$$\lambda^2 + \gamma\lambda + \omega^2 = 0.$$

Then we have

$$\begin{pmatrix} 0 & -\omega \\ \omega & -\gamma \end{pmatrix} \begin{pmatrix} 1 \\ -\lambda_1/\omega \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \omega + \gamma\lambda_1/\omega \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ -\lambda_1/\omega \end{pmatrix}$$

$$\begin{pmatrix} 0 & -\omega \\ \omega & -\gamma \end{pmatrix} \begin{pmatrix} 1 \\ -\lambda_2/\omega \end{pmatrix} = \begin{pmatrix} \lambda_2 \\ \omega + \gamma\lambda_2/\omega \end{pmatrix} = \lambda_2 \begin{pmatrix} 1 \\ -\lambda_2/\omega \end{pmatrix}$$

[You should probably check that this works.]

Our next step is to express the initial condition $\bar{u}(0) = (\omega x(0), -x'(0))$ in terms of the two eigenvectors.

$$\begin{pmatrix} \omega x(0) \\ -x'(0) \end{pmatrix} = \,?\begin{pmatrix} 1 \\ -\lambda_1/\omega \end{pmatrix} + \,?\begin{pmatrix} 1 \\ -\lambda_2/\omega \end{pmatrix}$$

How bout I just tell you the answer again?
For convenience we'll let

$$\Delta := \sqrt{\gamma^2 - 4\omega^2}$$

Then we have

$$\begin{pmatrix} \omega x(0) \\ -x'(0) \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -\lambda_1/\omega \end{pmatrix} + \beta \begin{pmatrix} 1 \\ -\lambda_2/\omega \end{pmatrix}$$

where $\alpha = \dfrac{\omega}{\Delta}\left[\dfrac{\gamma}{2} + \dfrac{1}{2}x(0) + x'(0)\right]$

and $\beta = \dfrac{\omega}{\Delta}\left[-\dfrac{\gamma}{2} + \dfrac{1}{2}x(0) - x'(0)\right]$.

[Yes, I used my computer for this.]

Finally, we compute the solution.

$$\begin{pmatrix} \omega x(t) \\ -x'(t) \end{pmatrix} = e^{At} \begin{pmatrix} \omega x(0) \\ -x'(0) \end{pmatrix}$$

$$= e^{At}\left[\alpha \begin{pmatrix} 1 \\ -\lambda_1/\omega \end{pmatrix} + \beta \begin{pmatrix} 1 \\ -\lambda_2/\omega \end{pmatrix}\right]$$

$$= \alpha e^{At} \begin{pmatrix} 1 \\ -\lambda_1/\omega \end{pmatrix} + \beta e^{At} \begin{pmatrix} 1 \\ -\lambda_2/\omega \end{pmatrix}$$

$$= \alpha e^{\lambda_1 t} \begin{pmatrix} 1 \\ -\lambda_1/\omega \end{pmatrix} + \beta e^{\lambda_1 t} \begin{pmatrix} 1 \\ -\lambda_2/\omega \end{pmatrix} .$$

Since the 2nd entries are redundant we delete them to get

$$\omega x(t) = \alpha e^{\lambda_1 t} + \beta e^{\lambda_2 t} .$$

By the Quadratic Formula we have

$$\lambda_1, \lambda_2 = (-\gamma \pm \Delta)/2 .$$

My computer simplified the answer to this:

$$x(t) =$$

$$\frac{1}{2\Delta} e^{-\gamma t/2} \Big[ \big(\gamma x(0) + \Delta x(0) + 2x'(0)\big) e^{\Delta t/2} + \big(-\gamma x(0) + \Delta x(0) - 2x'(0)\big) e^{-\Delta t/2} \Big]$$

Then I simplified it to this:

$$x(t) = \frac{1}{\Delta} e^{-\gamma t/2} \Big[ (\Delta x(0)) \cosh\left(\frac{\Delta}{2} t\right)$$
$$+ (\gamma x(0) + 2x'(0)) \sinh\left(\frac{\Delta}{2} t\right) \Big]$$

That doesn't look so bad, does it?

Now recall that

$$\Delta = \sqrt{\gamma^2 - 4\omega^2} = i\sqrt{4\omega^2 - \gamma^2}.$$

Let's define $\omega' := \sqrt{4\omega^2 - \gamma^2}\,/2$.

So that

$$x(t) = \frac{1}{2i\omega'} e^{-\gamma t/2} \Big[ (2i\omega' x(0)) \cosh(i\omega' t)$$
$$+ (\gamma x(0) + 2x'(0)) \sinh(i\omega' t) \Big]$$

Now recall that

$$\cosh(i\omega' t) = \cos(\omega' t)$$
$$\sinh(i\omega' t) = i\sin(\omega' t).$$

Finally, we conclude that

$$x(t) =$$

$$e^{-\gamma t/2}\left[x(0)\cos(\omega' t) + \left(\frac{\gamma x(0) + 2x'(0)}{2\omega'}\right)\sin(\omega' t)\right]$$

That actually looks pretty good.

Now let's think:

Without friction we have $\gamma = 0$, and

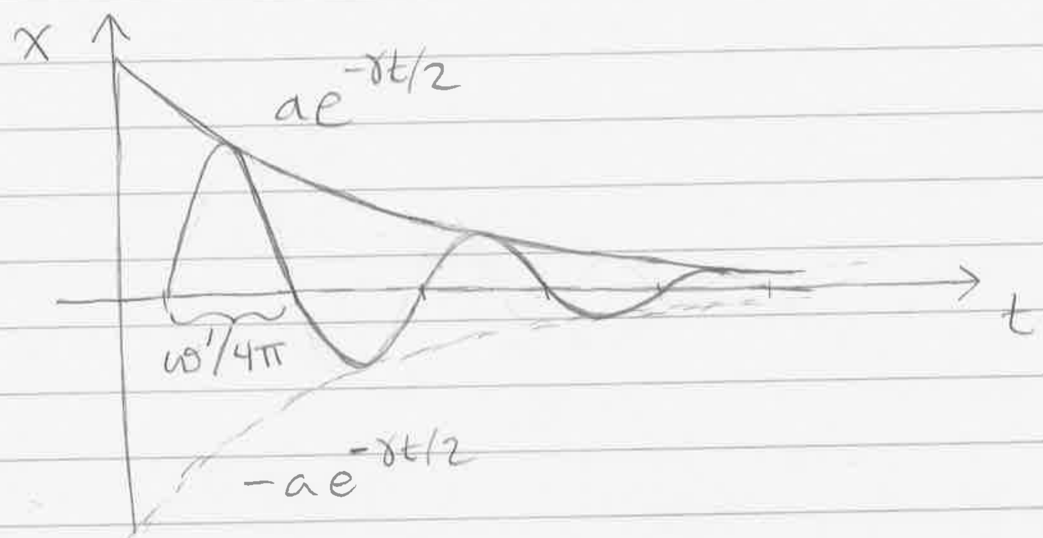
$$\omega' = \frac{1}{2}\sqrt{4\omega^2 - \gamma^2} = \frac{1}{2}\sqrt{4\omega^2} = \omega,$$

and we recover the old solution

$$x(t) = x(0)\cos(\omega t) + \frac{x'(0)}{\omega}\sin(\omega t).$$

Good. That suggests we didn't make any mistakes, or that our mistakes all canceled out.

With a little bit of friction $(4\omega^2 - \gamma^2 > 0)$, then $\omega'$ is a real number and we get a sine wave of frequency $\omega'$ bounded by a decaying exponential.




Quiz: What is a?

The other cases (too much friction, negative (!) friction, spring with negative (!) stiffness) are not so interesting.

2/24/15

HW 2 due Thurs Mar 5.
Quiz 3 Tues Mar 17
(after the Spring Break).

Recall: Last time we considered the "damped harmonic oscillator"

$$m x''(t) + \mu x'(t) + k x(t) = 0.$$

This models a particle of mass $m$ under the influence of a spring force of stiffness $k$ and a friction force of "viscosity" $\mu$.

The general solution is

$$x(t) = e^{-\gamma t/2}\left[x(0)\cos(\omega' t) + \left(\frac{\gamma x(0) + 2x'(0)}{2\omega'}\right)\sin(\omega' t)\right]$$

where $\gamma = \mu/m$, $\omega^2 = k/m$, and

$$\omega' = \frac{1}{2}\sqrt{4\omega^2 - \gamma^2}.$$

In the absence of friction ($\mu = 0$) we have

$$\gamma = 0 \text{ and } \omega' = \omega,$$

so that

$$x(t) = x(0)\cos(\omega t) + \frac{x'(0)}{\omega}\sin(\omega t).$$

This is the solution to the "undamped harmonic oscillator"

$$mx''(t) + kx(t) = 0. \quad ///$$

Example $(m=1, \mu=1, k=1)$:

$$x''(t) + x'(t) + x(t) = 0.$$

$\gamma = \mu/m = 1$, $\omega = k/m = 1$, and

$$\omega' = \frac{1}{2}\sqrt{4\omega^2 - \gamma^2} = \frac{1}{2}\sqrt{3}.$$

The general solution is

$$x(t) = e^{-t/2}\left[x(0)\cos\left(\frac{\sqrt{3}}{2}t\right) + \left(\frac{x(0)+2x'(0)}{\sqrt{3}}\right)\sin\left(\frac{\sqrt{3}}{2}t\right)\right]$$

For simplicity, assume that the particle begins at rest $(x'(0) = 0)$, so that

$$x(t) = e^{-t/2}\left[x(0)\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{x(0)}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)\right]$$

Can we graph this?

Recall that in general we have

$$a\cos(\omega t) + b\sin(\omega t) =$$

$$\sqrt{a^2+b^2}\cos\left(\omega t + \tan^{-1}\left(-\frac{b}{a}\right)\right).$$

In our case,

$$x(0)\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{x(0)}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)$$

$$= \sqrt{x(0)^2 + \frac{x(0)^2}{3}}\cos\left(\frac{\sqrt{3}}{2}t + \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right)$$

$$= \frac{2}{\sqrt{3}}x(0)\cdot\cos\left(\frac{\sqrt{3}}{2}t - \frac{\pi}{6}\right).$$

Hence

$$x(t) = \frac{2x(0)}{\sqrt{3}}e^{-t/2}\cdot\cos\left(\frac{\sqrt{3}}{2}t - \frac{\pi}{6}\right).$$

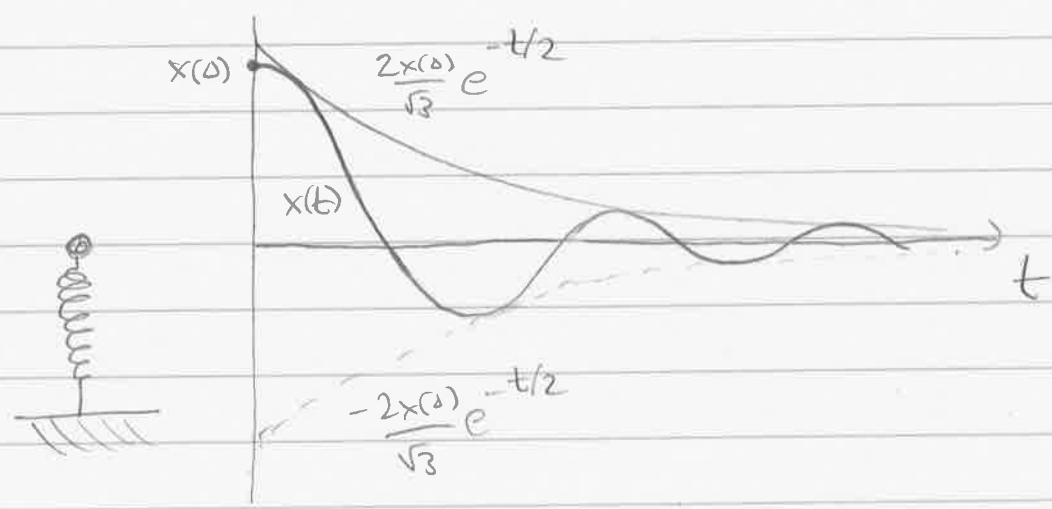
Let's check that $x(0)$ is correct:

$$x(0) = \frac{2x(0)}{\sqrt{3}} e^{0} \cdot \cos\left(-\frac{\pi}{6}\right)$$

$$= \frac{2x(0)}{\sqrt{3}} \cdot 1 \cdot \frac{\sqrt{3}}{2} = x(0) \quad \checkmark$$

This means we probably didn't make a mistake.

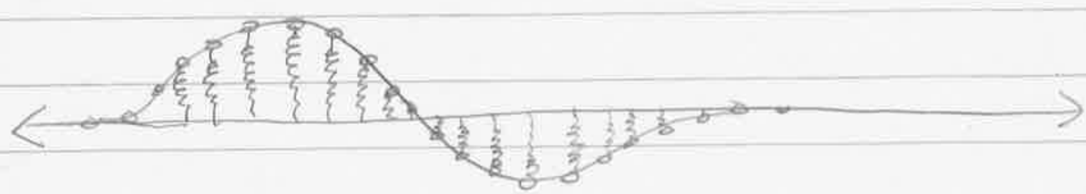
Now can we graph it? Yes.




[On HW 2 you will investigate the locations of local max/min, zeros, where the curves touch, etc.]

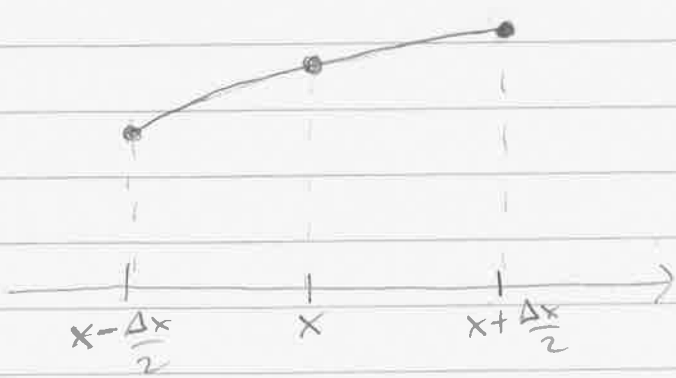
Back to music.

Now we are interested in a vibrating string of linear density $\rho$ under tension $T$. We can think of the string as an infinite collection of point masses, each of them acting as a harmonic oscillator:



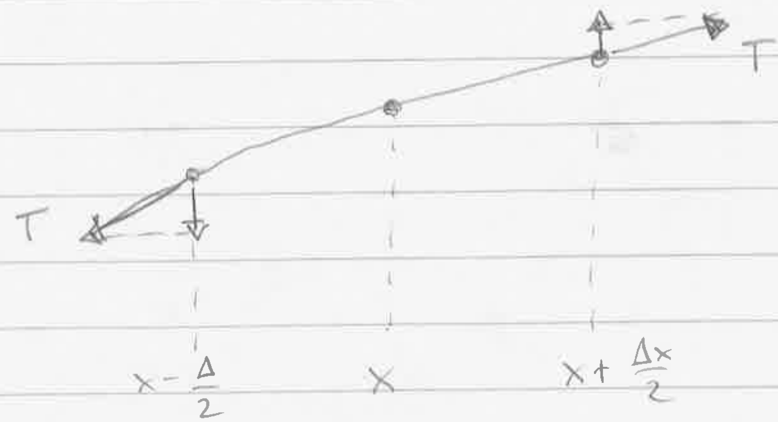
Let $u(x,t)$ be the vertical displacement of the string at time $t$ and position $x$.

Consider a very small interval $[x - \frac{\Delta x}{2}, x + \frac{\Delta x}{2}]$
Above this is a small piece of string with mass $\approx \rho \Delta x$ (density · length)



This piece of string can only move vertically, not horizontally. What force causes the motion?

Answer: The <u>tension</u> causes vertical motion

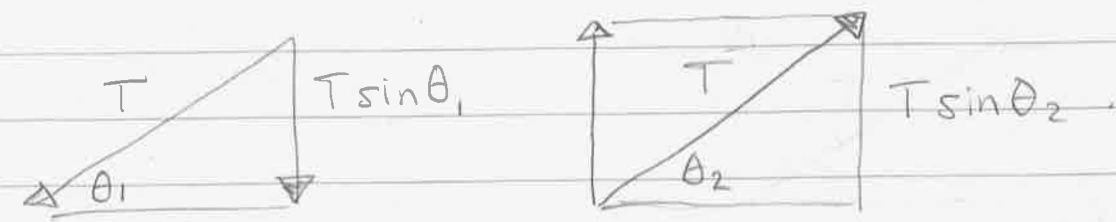



We get two little right triangles


The <u>slope</u> of the hypotenuse in each is

$$\tan(\theta_1) = \frac{d}{dx} u\left(x - \frac{\Delta}{2}, t\right), \quad \tan(\theta_2) = \frac{d}{dx} u\left(x + \frac{\Delta x}{2}, t\right).$$

Now we can compute the vertical components of the force.




Thus the total vertical vertical force at $x$ is approximately

$$T\sin\theta_2 - T\sin\theta_1$$

Using Newton's 2nd Law gives

$$\rho\,\Delta x\left(\frac{d}{dt}\right)^2 u(x,t) = T\sin\theta_2 - T\sin\theta_1 .$$

mass · acceleration = force

Finally, we assume that the angular displacement is small, so that

$$\sin\theta \approx \tan\theta$$

[Witchcraft!]

This gives

$$\rho \Delta x \left(\frac{d}{dt}\right)^2 u(x,t) \approx T \tan\theta_2 - T\tan\theta_1$$

$$= T\frac{d}{dx}u(x+\tfrac{\Delta}{2},t) - T\frac{d}{dx}u(x-\tfrac{\Delta}{2},t)$$

$$\Longrightarrow \quad \frac{\rho}{T}\left(\frac{d}{dt}\right)^2 u(x,t) = \frac{\frac{d}{dx}u(x+\frac{\Delta}{2},t) - \frac{d}{dx}u(x-\frac{\Delta}{2},t)}{\Delta x}$$

Taking $\Delta x \to 0$ gives

$$\frac{\rho}{T}\left(\frac{d}{dt}\right)^2 u(x,t) = \left(\frac{d}{dx}\right)^2 u(x,t).$$

Or we can write it as

$$\boxed{\frac{\rho}{T}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}}$$

This is called the Wave Equation (W.E.). After preliminary work by Vincenzo Galilei, Marin Mersenne and Brook Taylor, the equation was first written down by Jean le Rond d'Alembert in 1747.

Discussion:

- Can the W.E. be solved?
- What kind of solution do we expect?
- "How many" initial conditions are needed?
  (infinitely many, probably)

D'Alembert used a clever trick to get a general solution:

Define $c^2 := T/\rho$ and rewrite the W.E. as

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}\right) u = 0$$

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u = 0.$$

[Witchcraft!]

2/26/15

HW 2 due Thurs Mar 5
Spring Break Mar 9–13.

Last time we saw the Wave Equation (W.E.):

Consider an infinite thin string of linear density $\rho$ under tension $T$. If $u(x,t)$ is the vertical displacement of the string at position $x$ and time $t$ then we have

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}}$$

where $c^2 = T/\rho$. We gave a heuristic derivation of this equation assuming that the slope $\partial u/\partial x$ is never too large.

This equation was first written down by Jean le Rond d'Alembert (1747).

The W.E. is an example of a "partial differential equation" (p.d.e.). In general these are much harder to solve than o.d.e.'s.

Discussion:

- What would it mean to "solve" the W.E.?

- What kind of initial/boundary conditions are required to get a unique solution?

---

d'Alembert used a clever trick to find the general solution of the W.E.:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

$$\left( \frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} \right) u = 0.$$

Wait! What does this mean? We can think of the "thing"

$$\left( \frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} \right)$$

as an "operator" that sends functions to functions. In this case we are looking for "eigenfunctions" with eigenvalue zero.

In fact, this operator is <u>linear</u> in the sense that for all functions $u, v$ and constants $\alpha, \beta$ we have

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)(\alpha u + \beta v)$$

$$= \alpha\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)u + \beta\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)v.$$

So we can think of it as something like a matrix. Just as matrices can be composed/multiplied, so can operators. d'Alembert observed that

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right) = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right).$$

In what sense is this true?

Check: Let $u$ be a function. Then

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u$$

$$= \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x}\right)$$

$$= \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\frac{\partial u}{\partial t} - c\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\frac{\partial u}{\partial x}$$

$$= \frac{\partial^2 u}{\partial t^2} + c\,\cancel{\frac{\partial^2 u}{\partial x \partial t}} - c\,\cancel{\frac{\partial^2 u}{\partial t \partial x}} - c^2\frac{\partial^2 u}{\partial x^2}$$

$$= \frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right) u.$$

To get the result we had to assume that

$$\frac{\partial u}{\partial x \partial t} = \frac{\partial u}{\partial t \partial x}$$

"mixed partials commute"

This is only true if the function $u$ is "twice continuously differentiable". But we are thinking of $u(x,t)$ as the displacement of a physical string, so that's probably reasonable.

So we're looking for $u$ such that

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u = 0.$$

If we can solve $\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u = 0$ then that's good enough because then

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u$$
$$= \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) 0 = 0.$$

☆ Theorem (d'Alembert, 1747):

The W.E. has general solution

$$u(x,t) = f(x+ct) + g(x-ct)$$

where $f, g$ are <u>any</u> (sufficiently differentiable) functions $: \mathbb{R} \to \mathbb{R}$.

I won't prove that every solution has this form, but I will prove that this is a solution.

↓

Proof: Note that

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) f(x+ct)$$

$$= \frac{\partial}{\partial t} f(x+ct) - c\frac{\partial}{\partial x} f(x+ct)$$

$$= cf'(x+ct) - cf'(x+ct) = 0.$$

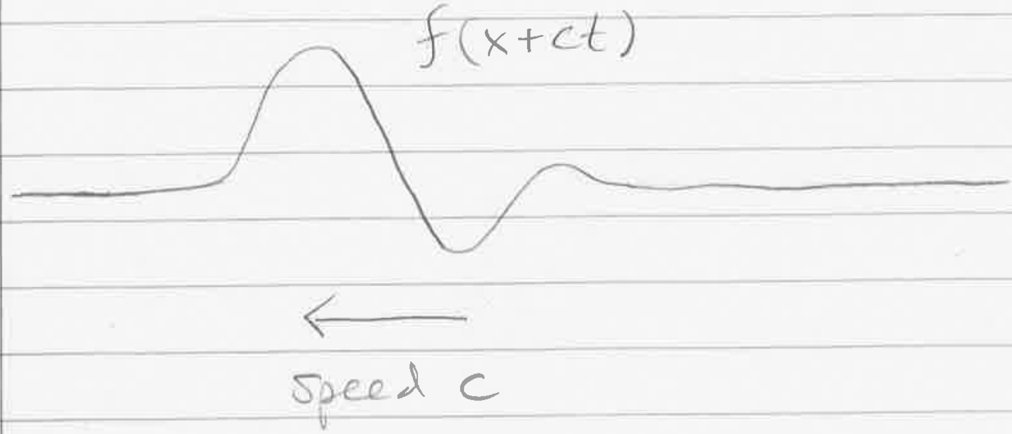
Similarly, $\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) g(x-ct) = 0.$

Since W.E. is linear we can add the two solutions. ///

We can think of $f(x+ct)$ and $g(x-ct)$ as arbitary disturbances traveling left and right, respectively, with constant speed

$$c = \sqrt{\frac{T}{\rho}}$$

This verifies Vincenzo's Observation. ✓




What happens when the disturbance hits a fixed boundary?

Let's assume that $u(0,t)=0 \ \forall t$. In other words

$$0 = f(ct) + g(-ct) \qquad \forall t$$

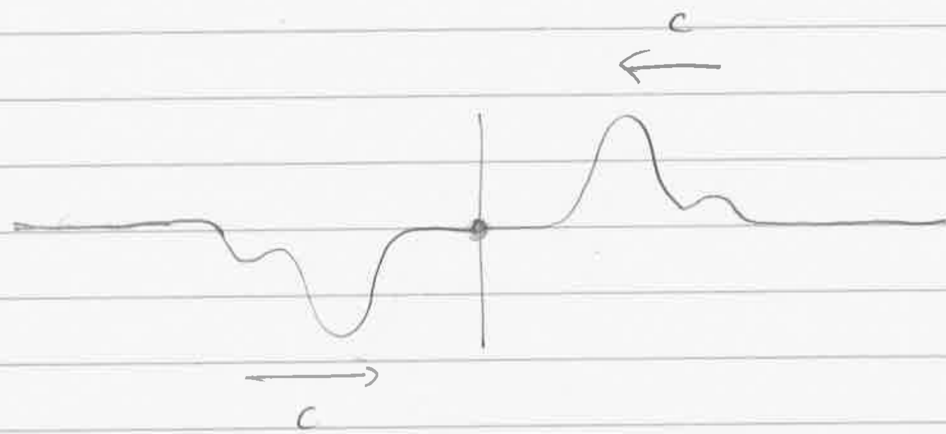
$$f(ct) = -g(-ct) \qquad \forall t$$

$$f(\lambda) = -g(-\lambda) \qquad \forall \lambda.$$

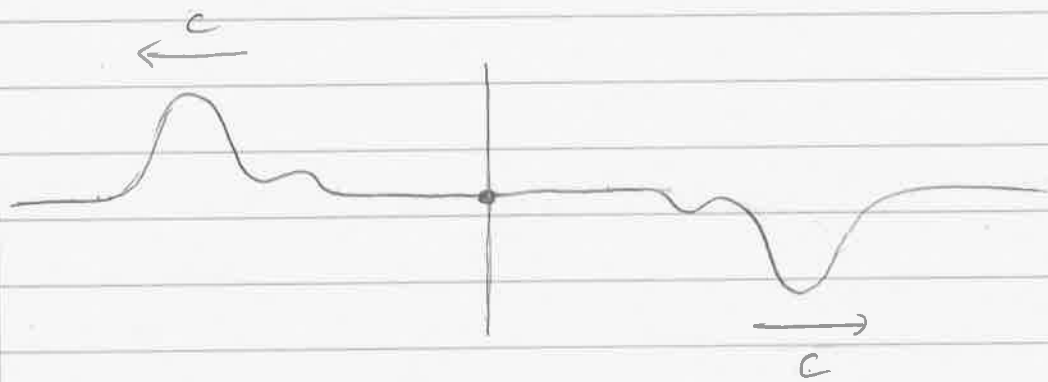
We conclude that

$$\begin{aligned} u(x,t) &= f(x+ct) + g(x-ct) \\ &= f(x+ct) - f(ct-x) \end{aligned}$$

This looks like: FIRST




THEN




From either side of the fixed boundary we see a wave reflected and rotated 180°.

This explains the Reflection Principle (R.P.) that we used earlier.

Finally, let's put another boundary at $x = \ell$. That is, let

$$u(\ell, t) = 0 \quad \forall t.$$

We have

$$u(x,t) = f(x+ct) - f(ct-x)$$

$$0 = f(\ell+ct) - f(ct-\ell) \qquad \forall t.$$
$$f(ct-\ell) = f(ct+\ell) \qquad \forall t.$$
$$f(ct-\ell) = f((ct-\ell)+2\ell) \qquad \forall t$$

$$f(\lambda) = f(\lambda+2\ell) \qquad \forall \lambda.$$

We conclude that $f$ must be periodic with period $2\ell$.

This explains our earlier calculation of the resonant frequency of a string of length $\ell$:

$$\frac{1}{2\ell}\sqrt{\frac{T}{\rho}}$$

Can you think of any functions with period $2\ell$ ?

Sure, how about

$$f(\lambda) = \cos\left(\frac{\pi n \lambda}{\ell} + \varphi\right), \quad n \in \mathbb{Z} \quad ?$$

This gives us a solution

$$u_n(x,t) = \cos\left(\frac{\pi n(x+ct)}{\ell} + \varphi\right) - \cos\left(\frac{\pi n(ct-x)}{\ell} + \varphi\right)$$

for any integer $n$

Exercise: We can rewrite this using a trig identity as

$$u_n(x,t) = 2\sin\left(\frac{\pi n x}{\ell}\right)\sin\left(\frac{\pi n c t}{\ell} + \varphi\right)$$

Finally, since the W.E. is linear, we can take any linear combination of these solutions to obtain...

☆ "Theorem" (Daniel Bernoulli, 1755):

The general solution of W.E. with fixed boundaries $u(0,t) = u(\ell,t) = 0 \ \forall t$ is

$$u(x,t) = \sum_{n \geq 0} \sin\left(\frac{\pi n x}{\ell}\right)\left[a_n \sin\left(\frac{\pi n c t}{\ell}\right) + b_n \cos\left(\frac{\pi n c t}{\ell}\right)\right]$$

for some constants $a_n, b_n \in \mathbb{R}$. ///

Apparently Euler did not believe that this is the <u>general</u> solution.

However, Bernoulli was later vindicated by the theory of Fourier Series.

[See the handout: "The vibrating string controversy" by Wheeler and Crummett.]

3/3/15

HW 2 due Thursday
Spring Break Mar 9 – 13
Quiz 3 Tues March 17.

Before we begin, here's a hint for HW 2 Problem 2.

"Galileo's Theory of Dissonance"

Let $a, b, A, B \in \mathbb{R}$ and consider

$$x(t) = A\sin(at) + B\cos(bt).$$

Show that $x(t)$ is periodic if and only if $a/b \in \mathbb{Q}$.

Hint: Suppose that $x(t)$ is periodic with period $T$, i.e.,

$$x(t+T) = x(t) \quad \forall\, t \in \mathbb{R}$$

Then we must also have

$$x''(t+T) = x''(t). \quad \forall\, t \in \mathbb{R}.$$

Compute

$$x''(t) = -Aa^2 \sin(at) - Bb^2 \cos(bt)$$

and observe that

$$x''(t) + b^2 x(t) = A(b^2 - a^2) \sin(at).$$

But we know that

$$x''(t+T) + b^2 x(t+T) = x''(t) + b^2 x(t)$$

for all $t \in \mathbb{R}$, and hence

$$\sin(a(t+T)) = \sin(at) \quad \forall\, t \in \mathbb{R}.$$

What do you conclude about $T$?

Galileo's Theory is an attempt to explain Pythagoras' Observation. He claims that we prefer small whole number ratios because we prefer periodic sounds (or at least sounds with small period). Non-periodic sounds keep the ear drum "in perpetual torment".

What do you think of Galileo's Theory?

It is largely discredited by experiment. (For example, strange frequency ratios with non-harmonic partials can be made to sound consonant. This suggests that von Helmholtz' Theory is more correct.)

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Recall: Last time we discussed the Wave Equation

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}}$$

OR

$$\boxed{u_{tt} = c^2 u_{xx}}$$

D'Alembert (1747) gave the general solution

$$u(x,t) = f(x+ct) + g(x-ct)$$

where $f, g$ are <u>any</u> (sufficiently differentiable) functions from $\mathbb{R}$ to $\mathbb{R}$.

After imposing the boundary conditions

$$u(0,t) = u(l,t) = 0 \quad \forall\, t,$$

we found that

$$u(x,t) = f(x+ct) - f(ct-x)$$

where $f$ is any (sufficiently differentiable) <u>periodic</u> function

$$f(\lambda + 2l) = f(\lambda) \qquad \forall\, \lambda.$$

By choosing the <u>specific</u> periodic function

$$f(\lambda) = \cos\left(\frac{n\pi\lambda}{l} + \varphi\right), \quad n \in \mathbb{Z}$$

we obtained Daniel Bernoulli's (1755) solution

$$u_n(x,t) = \cos\left(\frac{n\pi(x+ct)}{\ell} + \varphi\right) - \cos\left(\frac{n\pi(ct-x)}{\ell} + \varphi\right)$$

Using the trig identity

$$\cos(u) - \cos(v) = 2\sin\left(\frac{u-v}{2}\right)\sin\left(\frac{u+v}{2}\right)$$

gives

$$u_n(x,t) = 2\underbrace{\sin\left(\frac{n\pi x}{\ell}\right)}_{\text{no } t \text{ here}}\ \underbrace{\sin\left(\frac{n\pi ct}{\ell} + \varphi\right)}_{\text{no } x \text{ here}}$$

$$= \sin\left(\frac{n\pi x}{\ell}\right)\left[a_n \sin\left(\frac{n\pi ct}{\ell}\right) + b_n \cos\left(\frac{n\pi ct}{\ell}\right)\right]$$

where $a_n$, $b_n$ are any constants.

Bernoulli then stated that any "superposition" of these solutions also gives a valid solution.

$$u(x,t) = \sum_{n \geq 0} \sin\left(\frac{n\pi x}{\ell}\right)\left[a_n \sin\left(\frac{n\pi c t}{\ell}\right) + b_n \cos\left(\frac{n\pi c t}{\ell}\right)\right].$$

Of course, we see that this $u(x,t)$ <u>does</u> satisfy the W.E., but Euler and Lagrange believed that this is not the <u>full</u> solution. Euler considered it <u>absurd</u> to say that any motion of a vibrating string could be expressed as a trigonometric series.

But Euler was wrong. Today we recognize Bernoulli's solution as valid, and a precursor of "Fourier Analysis".

Before proceeding to Fourier Series, let me give another derivation of Bernoulli's solution, emphasizing the connection with the

harmonic oscillator.

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Let's assume at the outset that we are looking for a separable solution

$$u(x,t) = \underbrace{f(x)}_{\text{no } t} \cdot \underbrace{g(t)}_{\text{no } x} .$$

Then the W.E. says

$$u_{tt} = c^2 u_{xx}$$

$$f(x)\, g''(t) = c^2 f''(x)\, g(t)$$

$$\underbrace{c^2 \frac{g(t)}{g''(t)}}_{\text{no } x} = \underbrace{\frac{f(x)}{f''(x)}}_{\text{no } t}$$

Since the LHS doesn't involve $x$ and the RHS doesn't involve $t$ they must both be constant!

$$c^2 \frac{g(t)}{g''(t)} = \frac{f(x)}{f''(x)} = \lambda \in \mathbb{R}.$$

This gives us two o.d.e.'s :

$$\text{(1)} \qquad f''(x) = \lambda f(x)$$
$$\text{(2)} \qquad g''(t) = c^2 \lambda g(t) .$$

And we know how to solve these !

For (1) we have

$$f(x) = f(0)\cos\left(\sqrt{-\lambda}\, x\right) + \frac{f'(0)}{\sqrt{-\lambda}} \sin\left(\sqrt{-\lambda}\, x\right)$$

and for (2) we have

$$g(t) = g(0)\cos\left(c\sqrt{-\lambda}\, t\right) + \frac{g'(0)}{c\sqrt{-\lambda}} \sin\left(c\sqrt{-\lambda}\, t\right) .$$

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Imposing the boundary conditions

$$u(0,t) = u(\ell, t) = 0 \quad \forall t .$$

$$f(0)g(t) = f(\ell)g(t) = 0 \quad \forall t .$$

i.e., $f(0) = f(\ell) = 0$

gives

$$f(x) = \frac{f'(0)}{\sqrt{-\lambda}} \sin\left(\sqrt{-\lambda}\, x\right),$$

and then

$$0 = f(\ell) = \frac{f'(0)}{\sqrt{-\lambda}} \sin\left(\sqrt{-\lambda}\, \ell\right).$$

If $f(x) \not\equiv 0$ this implies

$$\sqrt{-\lambda}\, \ell = n\pi \quad \text{for some } n \in \mathbb{Z}$$

$$\sqrt{-\lambda} = \frac{n\pi}{\ell}$$

Finally, for each $n \in \mathbb{Z}$ we have a solution

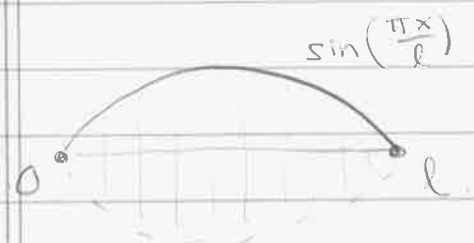
$$u_n(x,t) = f(x) \cdot g(t)$$

$$= \frac{f'(0)}{n\pi/\ell} \sin\left(\frac{n\pi x}{\ell}\right)\left[g(0)\cos\left(\frac{n\pi c t}{\ell}\right) + \frac{g'(0)}{n\pi c/\ell} \sin\left(\frac{n\pi c t}{\ell}\right)\right]$$

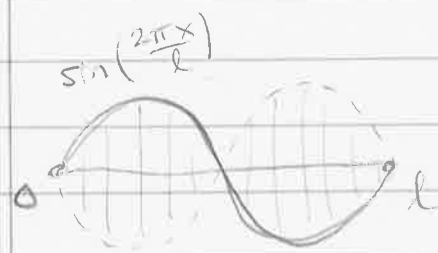
$$= \sin\left(\frac{n\pi x}{\ell}\right)\left[a_n \sin\left(\frac{n\pi c t}{\ell}\right) + b_n \cos\left(\frac{n\pi c t}{\ell}\right)\right]$$

for some $a_n, b_n \in \mathbb{Z}$.

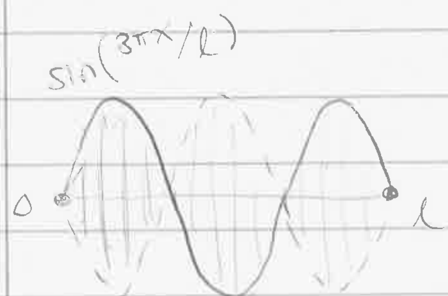
What does these solutions look like?




$u_1(x,t)$




$u_2(x,t)$




$u_3(x,t)$

They are just the "standing wave" solutions.

[ See Maple animation. ]

3/5/15

HW 2 due now.
Spring Break next week.
Quiz 3 Tues after Spring Break.

Today: Review for Quiz 3.

The solution of

$$x'(t) = a\,x(t)$$

is given by the convergent power series

$$x(0)\left[1 + (at) + \frac{(at)^2}{2} + \frac{(at)^3}{6} + \cdots + \frac{(at)^n}{n!} + \cdots\right]$$

which goes by the name

$$x(t) = x(0)\,e^{at}.$$

More generally, if $\overline{x}(t)$ is a parametrized curve in $\mathbb{R}^n$ and $A : \mathbb{R}^n \to \mathbb{R}^n$ is an $n \times n$ matrix encoding a vector field on $\mathbb{R}^n$, then the solution of

$$\overline{x}'(t) = A \cdot \overline{x}(t)$$

is given by the convergent power series

$$\left[ I + At + A^2\frac{t^2}{2} + A^3\frac{t^3}{6} + \cdots + A^n\frac{t^n}{n!} + \cdots \right] \cdot \bar{x}(0)$$

which goes by the name

$$\bar{x}(t) = e^{At} \cdot \bar{x}(0) .$$

To find the explicit solution we should compute the eigenvalues / eigenvectors of A and then express $\bar{x}(0)$ in terms of eigenvectors. Recall:

$$A\bar{x} = \lambda\bar{x} \implies e^{At} \cdot \bar{x} = e^{\lambda t} \cdot \bar{x} .$$

Example: Let $A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$. Solve

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} .$$

The eigenvalues of A are solutions of the characteristic equation

$$(1-\lambda)(1-\lambda) - 2(-2) = 0$$

$$\Downarrow$$

$$(1-\lambda)^2 = -4$$
$$1-\lambda = \pm 2i$$
$$\lambda = 1 \pm 2i \, .$$

Compute the eigenvectors:

$$\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = (1+2i)\begin{pmatrix} u \\ v \end{pmatrix}$$

$$u - 2v = (1+2i)u$$
$$\cancel{2u + v = (1+2i)v}$$

The 2nd equation is redundant so we can throw it away. Then choose $u=1$ to get

$$1 - 2v = 1+2i$$
$$-2v = 2i$$
$$v = -i$$

Check:

$$\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} 1+2i \\ 2-i \end{pmatrix} = (1+2i)\begin{pmatrix} 1 \\ -i \end{pmatrix} \checkmark$$

A similar calculation shows that

$$\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1-2i \\ 2+i \end{pmatrix} = (1-2i) \begin{pmatrix} 1 \\ i \end{pmatrix}$$

So the eigenvectors are $\begin{pmatrix} 1 \\ i \end{pmatrix}, \begin{pmatrix} 1 \\ -i \end{pmatrix}$

Our initial condition is $(x(0), y(0))$.

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ i \end{pmatrix} + \beta \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}^{-1} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} x(0) - \frac{1}{2} i\, y(0) \\ \frac{1}{2} x(0) + \frac{1}{2} i\, y(0) \end{pmatrix}.$$

The final solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{At} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}$$

$$= e^{At} \left[ \alpha \begin{pmatrix} 1 \\ i \end{pmatrix} + \beta \begin{pmatrix} 1 \\ -i \end{pmatrix} \right]$$

$$= \alpha e^{At} \begin{pmatrix} 1 \\ i \end{pmatrix} + \beta e^{At} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$= \alpha e^{(1-2i)t} \begin{pmatrix} 1 \\ i \end{pmatrix} + \beta e^{(1+2i)t} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$= e^{t} \begin{pmatrix} \alpha e^{-2it} + \beta e^{2it} \\ \alpha i e^{-2it} - \beta i e^{2it} \end{pmatrix}$$

Then plug in $\alpha, \beta = \frac{1}{2} x(0) \pm \frac{i}{2} y(0)$ and simplify to get

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{t} \begin{pmatrix} x(0) \cos(2t) - y(0) \sin(2t) \\ x(0) \sin(2t) + y(0) \cos(2t) \end{pmatrix}.$$

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Was there an easier way? Yes. Note that

$$A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$$

[ It is a fact that

$$e^{B+C} = e^B e^C \iff BC = CB. \ ]$$

Since $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

we have

$$e^{At} = e^{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}t + \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}t}$$

$$= e^{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}t} \cdot e^{\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}t}$$

$$= \begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix}\begin{pmatrix} \cos(2t) & -\sin(2t) \\ \sin(2t) & \cos(2t) \end{pmatrix}$$

$$= e^t \begin{pmatrix} \cos(2t) & -\sin(2t) \\ \sin(2t) & \cos(2t) \end{pmatrix}.$$

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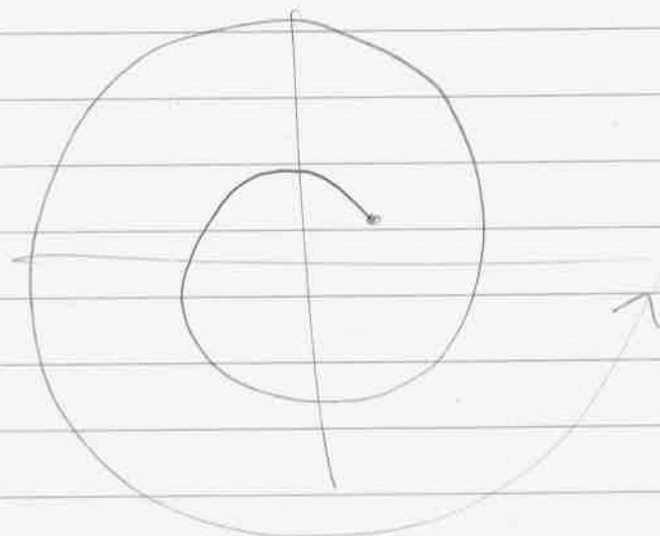
The general solution looks like



Example: Damped Harmonic Oscillator

$$x''(t) + \gamma x'(t) + \omega^2 x(t) = 0.$$

Define $\omega' = \frac{1}{2}\sqrt{4\omega^2 - \gamma^2}$. We can make a very clever choice of coordinates:

$$u_1(t) := x(t)$$

$$u_2(t) := \frac{-1}{2\omega'}\left(\gamma x(t) + 2x'(t)\right).$$

Then we have [check]

$$u_1'(t) = x'(t) = -\frac{\gamma}{2} u_1(t) - \omega' u_2(t)$$

$$u_2'(t) = \frac{1}{2\omega'}\left(\gamma x'(t) + 2x''(t)\right) = \omega' u_1(t) - \frac{\gamma}{2} u_2(t).$$

so that

$$\begin{pmatrix} u_1'(t) \\ u_2'(t) \end{pmatrix} = \begin{pmatrix} -\gamma/2 & -\omega' \\ \omega' & -\gamma/2 \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$$

Thus the solution is

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = e^{\begin{pmatrix} -\gamma/2 & 0 \\ 0 & \gamma/2 \end{pmatrix} t + \begin{pmatrix} 0 & -\omega' \\ \omega' & 0 \end{pmatrix} t} \cdot \begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix}$$

$$= \begin{pmatrix} e^{-\gamma t/2} & 0 \\ 0 & e^{-\gamma t/2} \end{pmatrix} \begin{pmatrix} \cos\omega' t & -\sin\omega' t \\ \sin\omega' t & \cos\omega' t \end{pmatrix} \begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix}.$$

$$\Longrightarrow \quad x(t) = u_1(t)$$

$$= e^{-\gamma t/2}\left[u_1(0)\cos(\omega' t) - u_2(0)\sin(\omega' t)\right]$$

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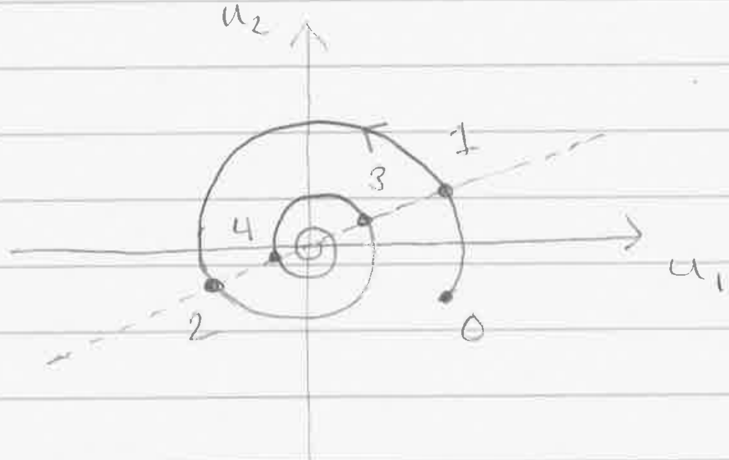
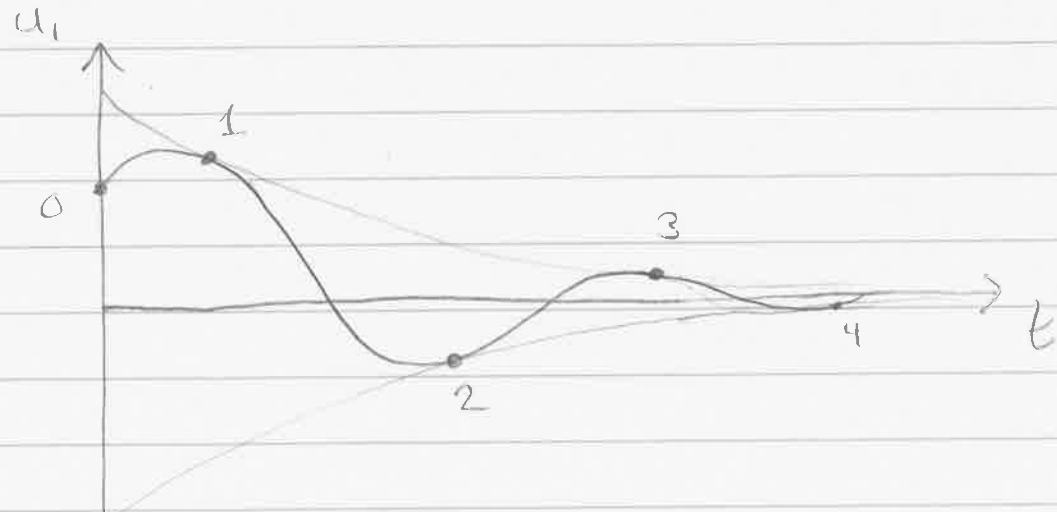
If $\gamma > 0$ and $\omega' \in \mathbb{R}$ (i.e. $4\omega^2 - \gamma^2 > 0$) then the general solution in phase space looks like




The graph of $u_1(t) = x(t)$ looks like

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Finally, recall that any function of the form

$$x(t) = a\cos(\omega t) + b\sin(\omega t)$$

can be expressed as

$$x(t) = r\cos(\omega t + \varphi).$$

Q: How are $a, b$ related to $r, \varphi$?

A: Use the angle sum identity

$$x(t) = \underbrace{r\cos\varphi}_{a}\cos(\omega t) \underbrace{- r\sin\varphi}_{b}\sin(\omega t)$$

Then we have

$$a = r\cos\varphi \qquad r = \sqrt{a^2+b^2}$$
$$b = -r\sin\varphi \qquad \varphi = \tan^{-1}(-b/a)$$

Q: What can you say about

$$x(t) = a\cos(\omega t) + b\sin(\mu t)\ ?$$