4/9/15

Quiz 4 next Tues Apr 14
(please study)
HW3 will follow.

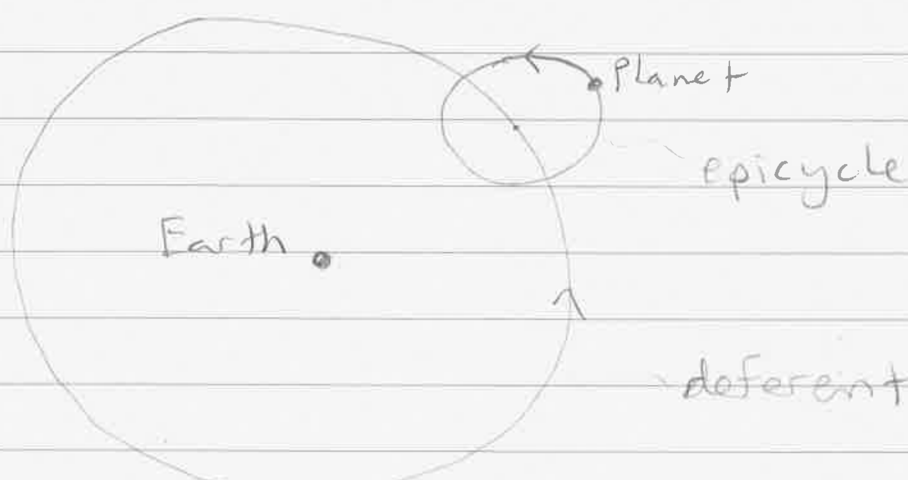
We have finished discussing Fourier series and the "principle of superposition". This completes our discussion of Pythagoras' Observation.

In the remaining time we will discuss some less mathematical aspects of the mathematics of music.

Topic: Scales and Temperaments.

Segue (Fourier series $\rightsquigarrow$ Scales):

In Ptolemy's "Almagest" (c. 150 AD) he modeled the orbits of planets with <u>deferents</u> and <u>epicycles</u>



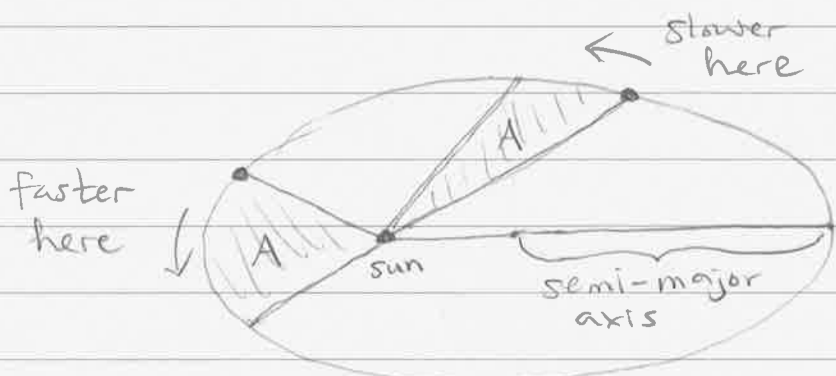
This become the dominant model of astronomy until it was overturned by Johannes Kepler around 1609.

Kepler's Laws of Planetary Motion:

1. The orbit of a planet is an <u>ellipse</u> with the sun at one focus.

2. A line segment connecting a planet to the sun sweeps out equal areas in equal times

3. The square of the period is proportional to the cube of the semi-major axis.



Kepler's Laws soon led to Newton's Law of Universal Gravitation (~1670).

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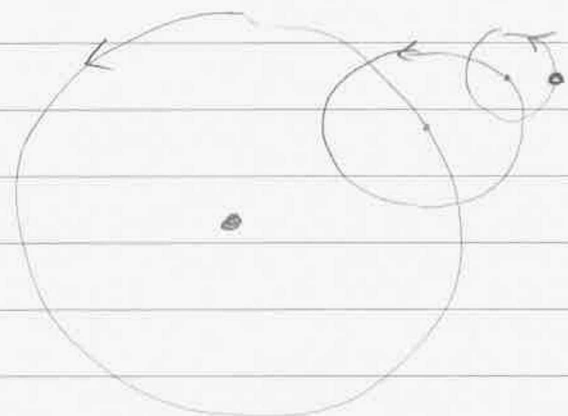
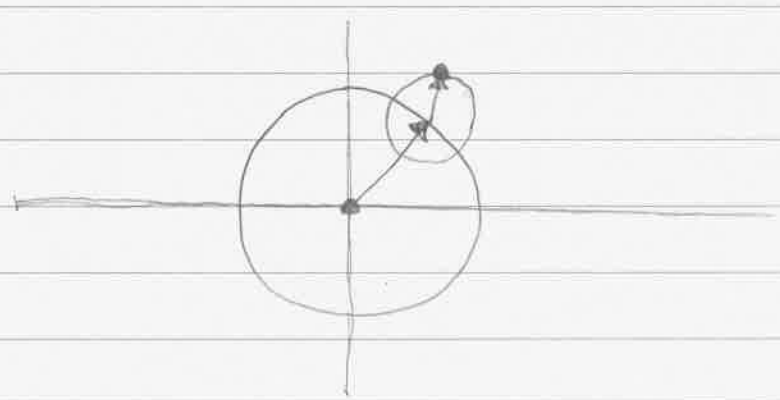
By Kepler's time, astronomers needed to use epicycles upon epicycles to keep up with newer, more accurate, observations. Different systems put the sun or the earth at the center.



It got very complicated, but somehow it always worked. Why?

Let's express the epicycles in coordinates. Put the sun (or earth) at the center of the complex plane.

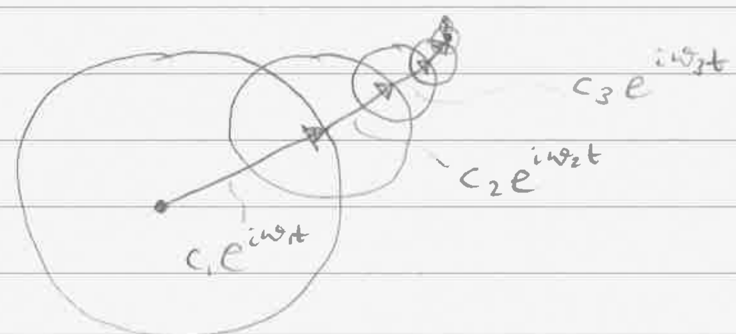


Suppose the deferent has radius c_1 and the epicycle has radius c_2. Suppose the center of of epicycle rotates around O with frequency ω_1 and the epicycle itself rotates with frequency ω_2.

Then the position of the planet at time t is

$$z(t) = c_1 e^{i\omega_1 t} + c_2 e^{i\omega_2 t}.$$

We can add as many epicycles as we want



The position of the planet will be

$$z(t) = \sum_{n \geq 1} c_n e^{i\omega_n t}$$

Can <u>any</u> planetary motion be modeled this way?

Assume the motion is <u>periodic</u> with frequency ω. The each epicycle must have frequency $n\omega$ for some $n \in \mathbb{Z}$ (say $n<0$ is clockwise and $n>0$ is counterclockwise), so that

$$z(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t}$$

$$= c_0 + \sum_{n \geq 1} \left(c_n e^{in\omega t} + c_{-n} e^{-in\omega t} \right)$$

the average position of the planet

If we define $a_n := c_n + c_{-n}$ and $b_n := i(c_n - c_{-n})$ then by applying Euler's formula we get

$$z(t) = a_0 \sum_{n \geq 1} a_n \cos(n\omega t) + b_n \sin(n\omega t)$$

The theory of Fourier series tells us that <u>any</u> (reasonable) periodic function can be represented in this way. After computing the coefficients a_n, b_n we can convert back to exponential form by

$$c_n = \begin{cases} a_0 & n = 0 \\ (a_n - ib_n)/2 & n > 0 \\ (a_{-n} + ib_{-n})/2 & n < 0 \end{cases}.$$

Thus any (reasonable) periodic orbit can be approximated by some configuration of epicycles.

This is why Ptolemy's theory was successful.

The reason I mention this now is because Ptolemy also studied music (in the "Harmonics") and he considered music and astronomy to be different aspects of the same subject, i.e., mathematics.

We have seen why the octave is divided into twelve intervals (to approximate perfect fifths).

But most melodies divide the octave into seven intervals, with the octave being the eighth note (hence "octave").

The names "perfect fourth" for the ratio 4:3 and "perfect fifth" for the ratio 3:2 are explained because these are the ratios from the first note to the fourth and fifth notes in the sequence, respectively.

Typically a division of the octave into seven intervals for melodic purposes is called a musical <u>SCALE</u>.

Q: Why are there 7 notes in a scale?

A: I don't know. Maybe because there are 7 planets.

In the Pythagorean tradition the 7 notes of the scale were associated to the 7 planets. Plato (in the "Timaeus") lists them in order of distance:

1. Moon
2. Sun
3. Venus
4. Mercury
5. Mars
6. Jupiter
7. Saturn.

2. Mercury
3. Venus
4. Sun

Later authors used this order

Beyond these seven "planets" (wandering stars) is the sphere of fixed stars (the Zodiac)

An early Pythagorean scale was described by Pliny (1st century AD) as follows:

- The Earth-Moon distance is 126,000 stadia, where 1 stadium is 625 paces.

- One Earth-Moon distance is called a tone. It is the difference between a fifth and a fourth

$$1 \text{ tone} = \frac{1 \text{ fifth}}{1 \text{ fourth}} = \frac{3/2}{4/3} = \frac{9}{8}.$$

- The planetary distances are

Planet	Distance From Earth
Moon	1
Mercury	$1 + 1/2$
Venus	2
Sun	$3 + 1/2$
Mars	$4 + 1/2$
Jupiter	5
Saturn	$5 + 1/2$
Zodiac	6

- The Earth - Zodiac distance is 6 times the Earth - Moon distance, which is 1 octave.

- The Earth - Sun distance is a perfect fifth.

Does that make sense?

6 tones = 1 octave ?

$$\left(\frac{9}{8}\right)^6 = 2$$

$$9^6 = 8^6 \cdot 2$$

$$3^{12} = 2^{19}$$

7/2 tones = 1 perfect fifth ?

$$\left(\frac{9}{8}\right)^{7/2} = \frac{3}{2}$$

$$\left(\frac{9}{8}\right)^7 = \left(\frac{3}{2}\right)^2$$

$$3^{12} = 2^{19}$$

Yes. As long as we assume that

$$3^{12} = 2^{19}$$

which the Pythagoreans pretty much did. ///

Remarks:

- This scale actually divides the octave into 8 because it includes the Zodiac/Earth. Later scales didn't

- Their association of pitch with height was the opposite of ours. Earth had the highest pitch, Zodiac had the lowest pitch.

[Listen to the scale]

4/14/15

Quiz 4 now (20 minutes)
HW 3 will be assigned tomorrow and
due on the last day of class (Apr 23).

Now: Scales & Temperaments.

The musical metric system.

In modern music we divide the octave into twelve equal intervals. Each such interval has a ratio of

$$(2/1)^{1/12} = 2^{1/12}/1.$$

and is called a semitone. An interval of two semitones is called a whole tone

$$\left(2^{1/12}/1\right)^2 = 2^{1/6}/1.$$

Thus, 6 whole tones make 1 octave. For finer measurements, we divide each semitone into 100 cents. Thus

$$1 \text{ cent} = \left(2^{1/12}/1\right)^{1/100}$$
$$= 2^{1/1200}/1.$$

and

$$1 \text{ octave} = 1200 \text{ cents}.$$

For whatever reason (maybe because of the 7 ancient planets) musical melodies often divide the octave into 7 notes called a scale.

There are many "colors" of scales. The most common scales are called diatonic. These divide the octave into

5 whole tones and
2 semitones

and the standard diatonic scale is called the major scale. Here is one notation for it:

do		re		mi		fa		so		la		ti		do
	T		T		S		T		T		T		S	

Technically, there are $\binom{7}{2} = 21$ possible diatonic scales but most of them are not used.

The 7 cyclic permutations of the major scale are called the classical "church modes". They have funny names:

Starting note	Name	
1	Ionian	(major).
2	Dorian	
3	Phrygian	
4	Lydian	
5	Mixolydian	(natural minor)
6	Aolian	
7	Locrian	

The second most popular scale is the Aolian mode, starting on the sixth note of the major scale

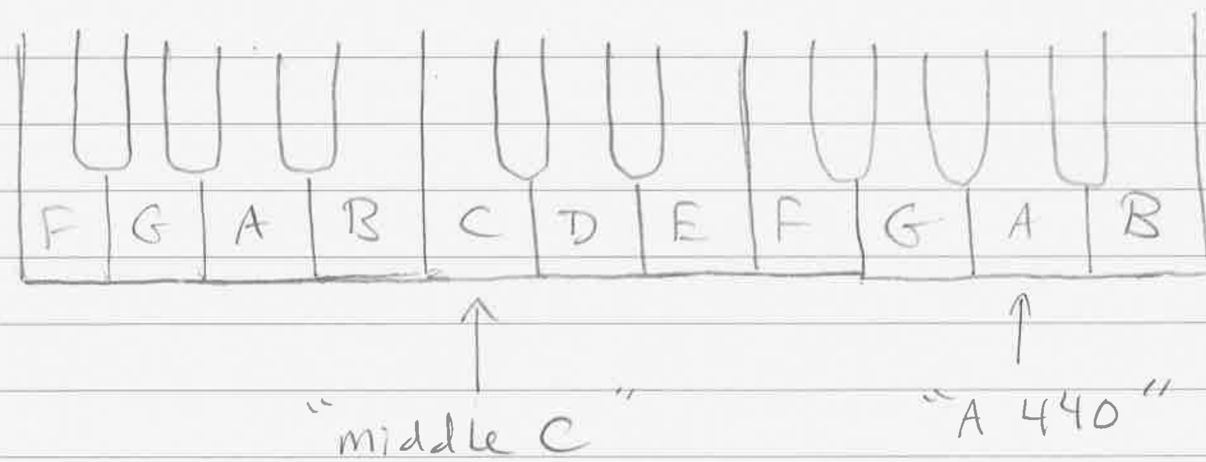
La ti do re mi fa so La
T S T T S T T

This is also called the natural minor scale. For whatever reason, the modern letter system for musical notes is based on this scale.

We label the notes with letters A through G and repeat periodically.

... F G A B C D E F G A B ...
T T T S T T S T T T S

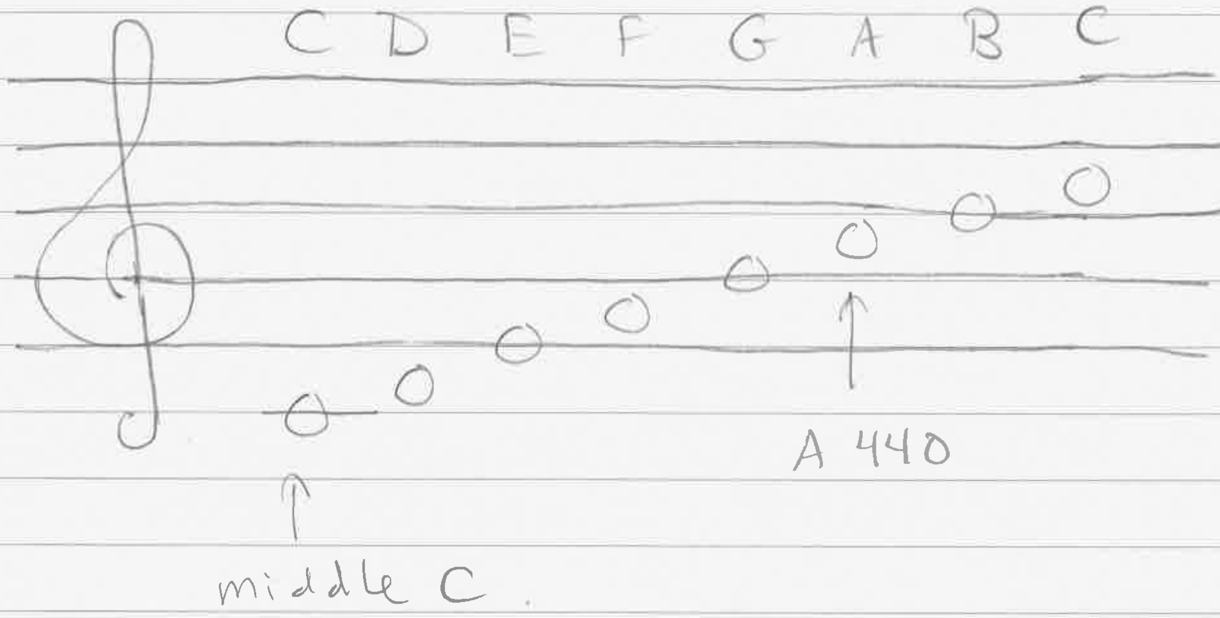
These are the white keys on the piano. Note that the semitones occur between BC and EF.



The most popular note is called "middle C", the first note (or the tonic) of the standard major scale.

But musical pitch is standardized to the A above middle C. It is taken to have fundamental frequency 440 Hz.

The most common way to notate music is using the treble clef (also called the "G clef").



Remark: The treble clef symbol looks like a G and it is centered on the G line.

Exercise: What is the pitch of middle C?

Remarks: Diatonic scales go back at least to ancient Greece. (Later we will discuss "Ptolemy's intense diatonic scale", a specific tuning of the major scale.) Modern music theory is also based on chords (notes played simultaneously) but we won't have time to discuss those.

As we discussed earlier, the modern system of "equal temperament" is a compromise.

It gives a very good approximation of perfect fifths

$$7 \text{ semitones} = 2^{7/12}/1 \approx 3/2$$

and it is flexible because it does not prefer any scale or key over another.

<u>But</u>, certain intervals are less than perfect. Recall the Plomp-Levelt dissonance curve

[see handout].

We observe that the equal-tempered fourth and fifth are close to their Pythagorean values of 4/3 and 3/2.

But some approximations are not so good. For example the equal-tempered "major third" (4 semitones) is noticeably "sharper" than its ideal value 5/4.

Q: By how much?

A: 4 equal tempered semitones is

$$\left(2^{1/12}\right)^4 = 2^{4/12} = 2^{1/3}$$

The difference from 5/4 is

$$\frac{2^{1/3}}{5/4} = \frac{2^{1/3} \cdot 4}{5} = \frac{2^{2+1/3}}{5} = \frac{2^{7/3}}{5}$$

How many cents is this?

$$\left(2^{1/1200}\right)^x = 2^{7/3}/5 .$$

Take logs to get

$$\frac{x}{1200} \log 2 = \frac{7}{3} \log 2 - \log 5$$

$$x = 1200 \left(\frac{7}{3} - \frac{\log 5}{\log 2} \right).$$

$$x = 13.7 \text{ cents} .$$

People say this is "easily audible".

///

Let's compare to the fourth and fifth.

Fourth: $\left(2^{1/1200}\right)^x = 2^{5/12} / \frac{4}{3} = 3 / 2^{19/12}$

$$\Longrightarrow \frac{x}{1200} \log 2 = \log 3 - \frac{19}{12} \log 2$$

$$\Longrightarrow x = 1200 \left(\frac{\log 3}{\log 2} - \frac{19}{12} \right)$$

$$x = 1.95 \text{ cents.}$$

Fifth: $\left(2^{1/1200}\right)^x = 2^{7/12} / \frac{3}{2} = 2^{19/12} / 3$

$$\Longrightarrow \frac{x}{1200} \log 2 = \frac{19}{12} \log 2 - \log 3$$

$$\Longrightarrow x = 1200 \left(\frac{19}{12} - \frac{\log 3}{\log 2} \right)$$

$$x = -1.95 \text{ cents.}$$

Hey, wait a minute. Why are those the same? Because the fourth and fifth are "dual" inside the octave:

$$\frac{4}{3} \cdot \frac{3}{2} = 2$$

1 fourth + 1 fifth = 1 octave. ///

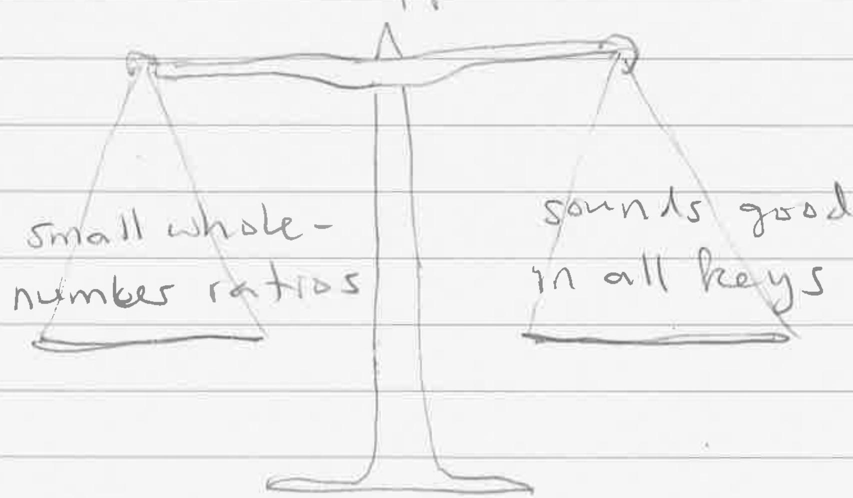
4/16/15

HW3 due Tues Apr 28
Last day of class Thurs Apr 24
NO FINAL EXAM

Last time we established the basics of musical notation and discussed the most commonly used scales.

Now we will discuss <u>tuning</u>, i.e., choosing exact pitches for the notes of a scale. The main problem is to reconcile Pythagoras' Observation that small whole number ratios sound good with the need for a flexible system that can handle complicated music with key changes.

You will see on HW3 that these two desires are in opposition:



There were four main phases in the Western history of musical tuning:

1) Pythagorean Tuning.
   - Based on 2s and 3s.

2.) Just Intonation
   - Base on small whole number ratios
   - Improves thirds and sixes from their Pythagorean values

3.) Meantone Temperament
   - Complicated attempts to tame the "wolf intervals" by carefully deviating from just intonation.

4.) Equal Temperament
   - Logical endpoint of meantone: all semitones are tuned to $2^{1/12}$.
   - Mathematical and technical advances were required to make this feasible. (The difficulty, believe it or not, was calculating an accurate value for $2^{1/12}$.)

[5.) Modern Academic Music
   - Experiments with other systems have not caught on outside academia.]

(1) Pythagorean Tuning.

Pythagorean harmony was based on the ratios 2/1 (octave) and 3/2 (perfect fifth).

In the "Timaeus", Plato describes the creation of the world in terms of the sequence

$$1, 2, 3, 4, 8, 9, 27$$

(powers of 2 & 3)

He describes these as the distances above earth of the 7 planetary spheres. If we extend the sequence to

$$1, 2, 3, 4, 8, 9, 16, 27, 32, 64, 81, 128, 243, 256$$

then the successive ratios give all the values of the Pythagorean musical intervals.

2/1	octave	8/4	octave
3/2	fifth	9/8	tone
4/3	fourth	16/9	minor seventh

27/16	major sixth	512/256	octave
32/27	minor third	729/512	tritone
64/32	octave	1024/729	?
81/64	major third		
128/81	minor sixth		
243/128	major seventh		
256/243	semitone.		

Let's Stop

Thus, the Pythagorean chromatic scale (i.e. the 12 tone scale) is

Note	Interval From 0th Note
0	1
1	256/243
2	9/8
3	32/27
4	81/64
5	4/3
6	729/512
7	3/2
8	128/81
9	27/16
10	16/9
11	243/128
12	2

The Pythagorean major scale is

Note	Interval From 0th Note
0	1
1	9/8
2	81/64
3	4/3
4	3/2
5	27/16
6	243/128
7	2

[Listen]

There is another way to obtain these numbers called the <u>spiral</u> <u>of</u> <u>fifths</u>.

If we move by perfect fifths we will return to our original letter note after 12 steps. (Actually we will be ≈ 7 octaves above our starting note.)

Starting at D, it looks like this:

Why did I choose D?

$\times 2$ 1

$2/3$ D $3/2$

$\times 4$ G A

$(2/3)^2$ $(3/2)^2 \div 2$

C E

$\times 4$ $(2/3)^3$ F Circle of Fifths B $(3/2)^3 \div 2$

$\times 8$ $(2/3)^4$ B^b $F^\#$ $(3/2)^4 \div 4$

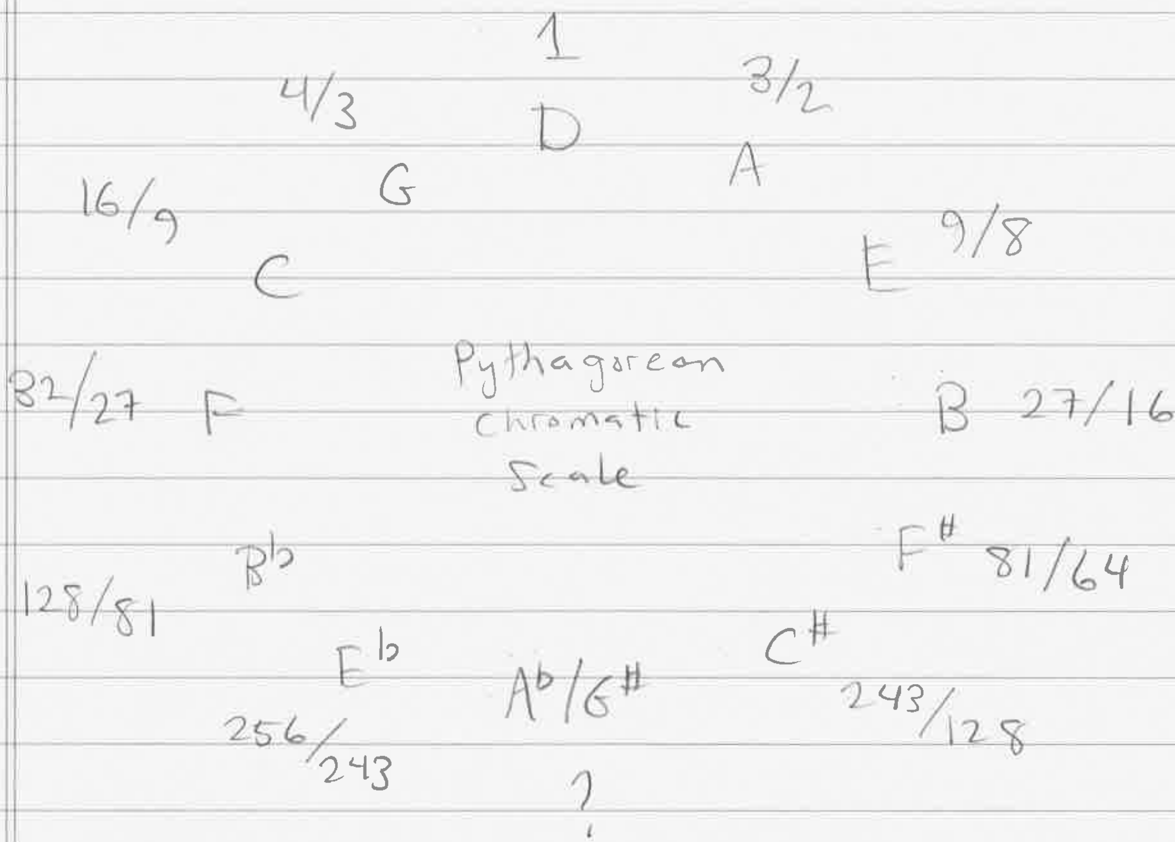
$\times 8$ $(2/3)^5$ E^b $A^b/G^\#$ $C^\#$ $(3/2)^5 \div 4$

?

Clockwise = multiply by 3/2
Counterclockwise = divide by 3/2.

To get the values of the Pythagorean scale we translate all of these into the same octave by multiplying or dividing by 2.

This gives the values :



What is the interval $D \to A^b/G^\#$?
There are two natural choices

$$A^b = (2/3)^6 \times 16 = 1024/729$$

$$G^\# = (3/2)^6 / 4 = 729/512 .$$

I guess we can choose 729/512 since the numbers are smaller. But in Pythagorean Tuning, A^b and $G^\#$ are really two <u>different</u> <u>notes</u>.

The difference between them

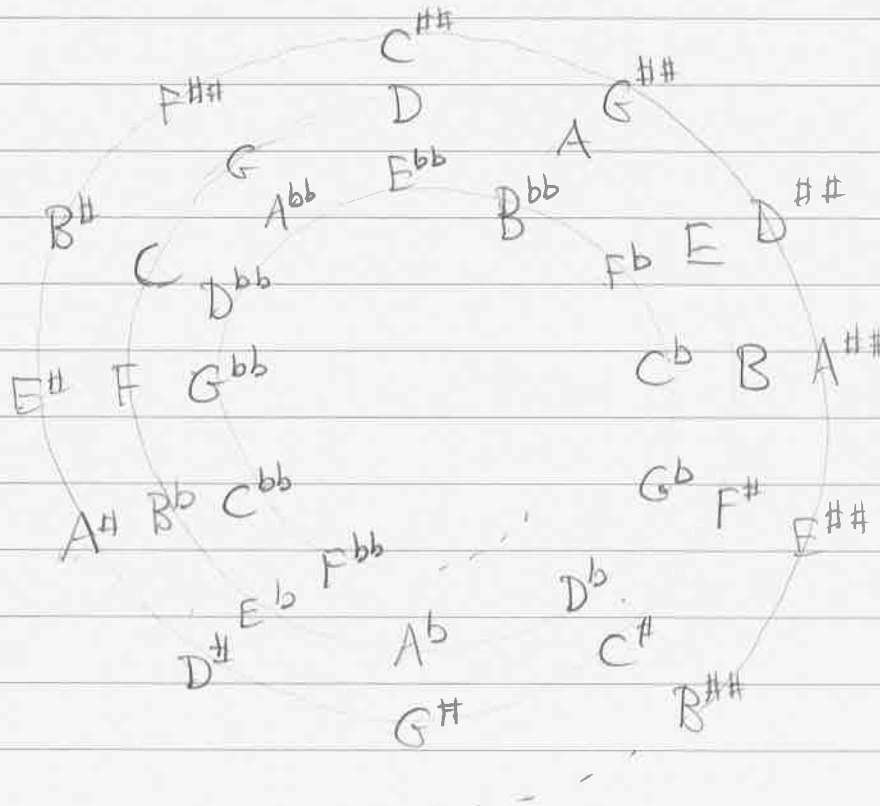
$$\frac{729/512}{1024/729} = \frac{3^{12}}{2^{19}} = 24.46 \text{ cents}$$

is called the Pythagorean comma.

It is audible.

So we should really draw the

Spiral of Fifths:



Obviously, you would need a 2-dimensional piano keyboard to achieve this.

But maybe we shouldn't bother:

- Ancient Greek music theory did not consider 3rds and 6ths to be consonant. Also, their ratios for 3rds and 6ths had large numbers, so they are less consonant than they could be. [I don't which is the chicken and which is the egg.]

- When 3rds and 6ths became popular (between 1300 and 1700 AD), the spirit of Pythagorean tuning (small whole numbers) was preserved, but the law of Pythagorean tuning (only use 2s and 3s) was thrown out.

Because of symmetry, the 3rds and 6ths can be improved with a single ratio

$$81/80 = 21.51 \text{ cents}$$

called the <u>syntonic</u> <u>comma</u>.

Here's the fix:

Interval	Pythagorean	Fix.	Just
Minor 3rd	32/27	+1 s.c.	6/5
Major 3rd	81/64	−1 s.c.	5/4.
Minor 6th	128/81	+1 s.c.	8/5
Major 6th	27/16	−1 s.c.	5/3.

The new system is called 5-limit tuning because it uses the primes 2, 3, 5.

The old (Pythagorean) system is called 3-limit tuning.

Q: If 5-limit is better than 3-limit, why not go to ∞-limit?

A: You will do that on HW 3.

4/21/15

Quiz 4 Results

Total : 10
Ave : 7.8
Med : 9
St. Dev : 2

HW 3 is due next Tues Apr 28.

Last time we discussed Pythagorean tuning which is based on the primes 2 & 3 and is sometimes called

3-limit tuning.

[The term "limit" in this sense was introduced by Harry Partch (1901-1974), an important figure in 20th century music theory.]

During the Renaissance, 3-limit tuning was extended to 5-limit, to obtain more consonant 3rds and 6ths.

Recall that the interval

$$81/80 = 21.51 \text{ cents}$$

is called a syntonic comma (or sometimes just a comma). Let's see how Pythagorean tuning can be improved by adding and subtracting syntonic commas.

Pythagorean	Fix.	Improved Tuning
1	—	1
256/243	+1 s.c.	16/15
9/8	—	9/8
32/27	+1 s.c.	6/5
81/64	−1 s.c.	5/4
4/3	—	4/3
729/512	−2 s.c.	25/18
3/2	—	3/2
128/81	+1 s.c.	8/5
27/16	−1 s.c.	5/3
16/9	+1 s.c.	9/5
243/128	−1 s.c.	15/8
2	—	2

The resulting major scalled is the modern justly tuned major scale:

do	1
re	9/8
mi	5/4
fa	4/3
so	3/2
la	5/3
ti	15/8
do	2.

This tuning goes back at least to Ptolemy (it was his syntonous or "intense diatonic scale", one of 8 that he proposed) but it was not popularized until the Renaissance. Gioseffo Zarlino (1517–1590) declared it is the only tuning that can be reasonably sung.

Traditionally, a just tuning is one obtained from Pythagorean tuning by adding or subtracting integer multiples of the syntonic comma. Thus every just tuning is 5-limit.

But 5-limit tuning seems rather arbitrary. Since the dissonance of a whole number ratio is governed mostly by the denominator we would like to solve the following problem.

☆ Problem of Just Tuning:

Given an interval $\alpha \in \mathbb{R}$ and an integer n we would like to find the fraction with denominator $\leq n$ that is closest to α. In other words we want to find $p, q \in \mathbb{Z}$ with $q \leq n$ such that

$$\left| \alpha - \frac{p}{q} \right|$$

is minimized. Such a fraction is called a <u>best</u> <u>rational</u> <u>approximation</u> to α. As $n \to \infty$ we obtain a sequence of best rational approximations converging to α.

The solution of this problem is given by "continued fractions".

Continued Fractions.

Given two integers a, b we can compute the greatest common divisor by repeatedly subtracting the smaller from the larger. This is called the Euclidean Algorithm.

Example: $a = 54$, $b = 25$.

a	b	$a-b$
54	25	29
29	25	4
25	4	21
21	4	17
17	4	13
13	4	9
9	4	5
5	4	1
4	1	3
3	1	2
2	1	1
1	1	0

this is the gcd.

We can make the algorithm quicker by subtracting a multiple of the smaller number.

a	b	a − multiple of b
54	25	$54 - 2 \cdot 25 = 4$.
25	4	$25 - 6 \cdot 4 = 1$
4	1	$4 - 4 \cdot 1 = 0$.

And we can express this algebraically by writing 54/25 as a continued fraction:

$$\frac{54}{25} = \frac{2 \cdot 25 + 4}{25}$$

$$= 2 + \frac{4}{25}$$

$$= 2 + \frac{1}{25/4}$$

$$= 2 + \frac{1}{(6 \cdot 4 + 1)/4}$$

$$= 2 + \cfrac{1}{6 + \cfrac{1}{4}}$$

DONE.

We can do the same thing even if a & b are "incommensurable".

Example : $a = \sqrt{2}$, $b = 1$.

$$\frac{\sqrt{2}}{1} = 1 + (\sqrt{2} - 1)$$

$$= 1 + \frac{1}{1/(\sqrt{2}-1)}$$

But note that

$$\frac{1}{\sqrt{2}-1} = \frac{1}{\sqrt{2}-1} \frac{(-\sqrt{2}-1)}{(-\sqrt{2}-1)}$$

$$= \frac{-\sqrt{2}-1}{-2+1} = \frac{-\sqrt{2}-1}{-1} = 1+\sqrt{2}$$

So that

$$\sqrt{2} = 1 + \frac{1}{1+\sqrt{2}}$$

After then the process repeats.

$$\sqrt{2} = 1 + \cfrac{1}{1+\sqrt{2}}$$

$$= 1 + \cfrac{1}{1 + \left(1 + \cfrac{1}{1+\sqrt{2}}\right)}$$

$$= 1 + \cfrac{1}{2 + \cfrac{1}{1+\sqrt{2}}}.$$

$$= 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}$$

forever.

This is called the "continued fraction expansion" of $\sqrt{2}$. For short we write

$$\sqrt{2} = [1; 2, 2, 2, 2, \ldots].$$

Here are the main properties of continued fractions.

☆ Theorem: Let $\alpha \in \mathbb{R}$ be irrational. Then there exists a unique sequence of integers $a_0, a_1, a_2, \cdots$ where $a_i > 0$ for all $i \geq 1$ and such that

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

We write this as $\alpha = [a_0; a_1, a_2, a_3, \cdots]$. The best rational approximations to α are then obtained as follows.

- Truncate the continued fraction to get

$$[a_0; a_1, a_2, \cdots, a_n].$$

- Replace a_n by an integer between $a_n/2$ and a_n.

- If a_n is even, $[a_0; a_1, a_2, \cdots, a_{n-1}, \frac{a_n}{2}]$ may or may not be a best rational approximation.

For example, the c.f.e. for π is

$$\pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, \cdots]$$

so the sequence of best rational approximations to π is

Approximation p/q	Error $\pi - p/q$
$[3] = 3$	$0.14159\cdots$
$[3;4] = 13/4$	$-0.10841\cdots$
$[3;5] = 16/5$	-0.05841
$[3;6] = 19/6$	-0.02507
$[3;7] = 22/7$	-0.00126
$[3;7,8] = 179/57$	0.00124
$[3;7,9] = 201/64$	0.00097
$[3;7,10] = 223/71$	0.00075
$[3;7,11] = 245/78$	0.00057
$[3;7,12] = 267/85$	0.00042
$[3;7,13] = 289/92$	0.00029
$[3;7,14] = 311/99$	0.00018
$[3;7,15] = 333/106$	0.00008
$[3;7,15,1] = 355/113$	-0.000000027

etc.

Remarks :

- The sign of the error switches every time we include a new term of the c.f.e.

- In a musical situation we would measure the error in cents. Below 10 cents is probably an inaudible difference.

- Therefore to get a good "just tuning" of α we would choose the first "best rational approximation" p/q such that

$$\left| 1200 \log\left(\frac{\alpha}{p/q}\right) / \log(2) \right| < 10 .$$

- On HW3 you will find the first b.r.a. within 50 cents of each note of the 12 tone equal tempered scale. This is the weakest restriction we can make without accidentally repeating notes.

4/23/15

HW 3 due next Tuesday.
 – under my door is OK.

Last Day of Class.

Recall: A just tuning system is obtained from Pythagorean tuning by adding and subtracting multiples of the syntonic comma

$$81/80 \approx 21.5 \text{ cents.}$$

Today we will discuss a notation for tuning that establishes a link with algebra.

To begin, we choose a base note, say $C = 1/1$, and use the letter names to denote Pythagorean tuning

Letter	⋯ Eb	Bb	F	C	G	D	A ⋯
Ratio	⋯ $8/27$	$4/9$	$2/3$	$1/1$	$3/2$	$9/4$	$27/8$ ⋯
Translated into first octave.	⋯ $32/27$	$16/9$	$4/3$	$1/1$	$3/2$	$9/8$	$27/16$

Because of the spiral of fifths, there are infinitely many notes, e.g.

$$729/512 = F\sharp \neq G\flat = 1024/729 .$$

Carl Eitz (1891) intoduced a superscript to indicate modifications by syntonic commas. Thus,

$$E\flat = 32/27$$

$$E\flat^{+1} = 32/27 \cdot 81/80 = 6/5$$

and

$$E = 81/64$$

$$E^{-1} = 81/64 \cdot 80/81 = 5/4 .$$

[We can use a superscript 0 to indicate the Pythagorean value.]

Now all the possible notes can be arranged in an infinite triangular lattice:

$\downarrow$

$$\begin{array}{ccccc} F\sharp^{-2} & C\sharp^{-2} & G\sharp^{-2} & D\sharp^{-2} & A\sharp^{-2} \end{array}$$

$$\begin{array}{cccccc} D^{-1} & A^{-1} & E^{-1} & B^{-1} & F\sharp^{-1} & C\sharp^{-1} \end{array}$$

$$\begin{array}{ccccccc} B\flat^{0} & F^{0} & C^{0} & G^{0} & D^{0} & A^{0} & E^{0} \end{array}$$

$$\begin{array}{cccccc} D\flat^{+1} & A\flat^{+1} & E\flat^{+1} & B\flat^{+1} & F^{+1} & C^{+1} \end{array}$$

$$\begin{array}{ccccc} F\flat^{+2} & C\flat^{+2} & G\flat^{+2} & D\flat^{+2} & A\flat^{+2} \end{array}$$

etc.

Q: Why did we do it this way?

A: So that a triangle $\triangle$ is a <u>just</u> <u>major</u> <u>triad</u>, e.g.,

$$\begin{array}{cc} \multicolumn{2}{c}{E^{-1}} \\ C^{0} & G^{0} \end{array} = \begin{array}{cc} \multicolumn{2}{c}{5/4} \\ 1/1 & 3/2 \end{array} = \begin{array}{cc} \multicolumn{2}{c}{5} \\ 4 & 6 \end{array}$$

This is called a 4:5:6 triad. It is the basis of Western harmony.

and a triangle ∇ is a <u>just</u> <u>minor</u> <u>triad</u>, e.g.,

$C^0 \quad G^0 = 1/1 \quad 3/2 = 10 \quad 15$

$\quad Eb^{+1} \qquad 6/5 \qquad 12$

The just minor triad has ratios $10:12:15$.

To play music in this system would require an infinite two-dimensional keyboard, which is not practical.

But many of the notes are indistinguishably close together. So we might as well truncate it somehow.

Q: How should we truncate it?

Observe that 3 perfect fifths + 1 minor third = 4 octaves + 1 sytonic comma:

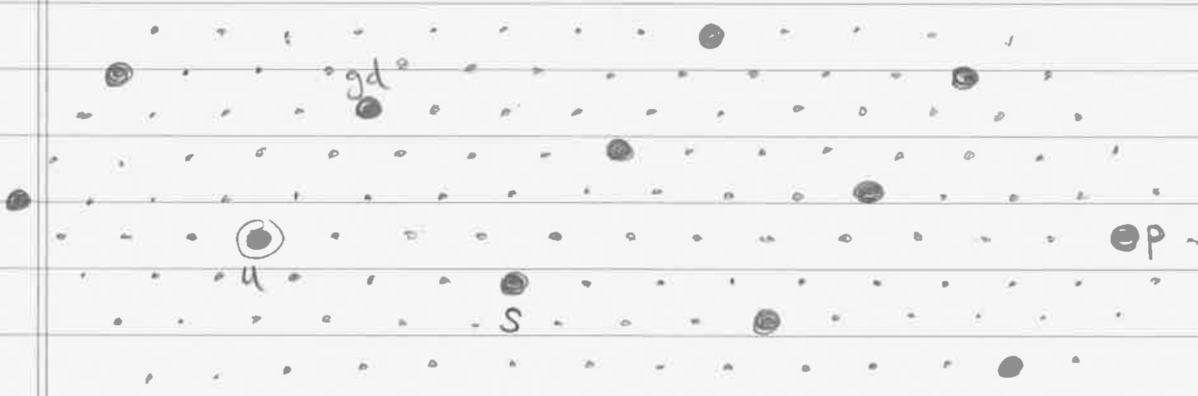
$$(3/2)^3 \cdot (6/5) = (2/1)^4 \cdot (81/80)$$

and 12 pefrect fifths = 7 octaves + 1 Pythagorean comma:

$$(3/2)^{12} = (2/1)^7 \cdot (3^{12}/2^{19}).$$

Thus it is reasonable to identify these notes in our infinite lattice. By doing so we end up identifying infinitely many notes; this is called the unison sublattice.

Here is a picture. All of the black dots are regarded to be "the same".



etc.

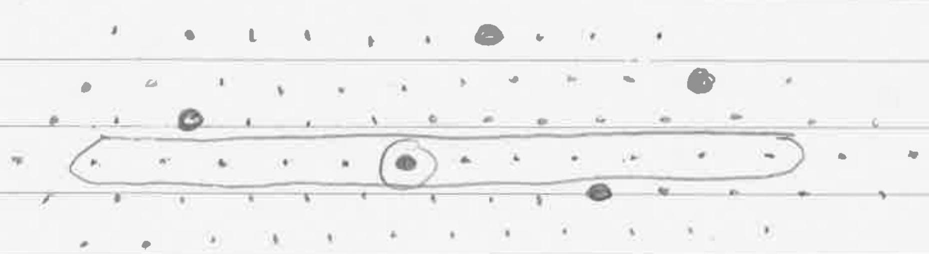
u = unison
s = syntonic comma
p = pythagorean comma
gd = greater diesis = p − 3·s

Since there are only 12 notes to the octave (one a one-dimensional piano), choosing a tuning system corresponds to choosing 12 points in the lattice. The only restriction is that they should be 12 <u>different</u> notes (i.e. no two can differ by a unison vector).

In algebraic language, a tuning system is just a set of coset representatives for the quotient group

lattice / unison lattice.

For example, Pythagorean tuning corresponds to these 12 points



or really any horizontal line of 12 notes.

But this is not very good for modern harmony because it doesn't contain <u>any</u> just major or minor triads.

Last time we discussed an improved tuning system. Here it is in Eitz notation.

Many people, including Euler (1739), would put the $F\sharp$ here instead.

$$\begin{array}{ccccccccc}
 & & F\sharp^{-2} & & & & & & \\
 & & & A^{-1} & E^{-1} & B^{-1} & F\sharp^{-1} & & \\
 & & F^{0} & C^{0} & G^{0} & D^{0} & & & \\
 & Db^{+1} & Ab^{+1} & Eb^{+1} & Bb^{+1} & & & &
\end{array}$$

This is better than Pythagorean because it contains lots of just triads.

There are infinitely many choices of coset representatives you can make, but none of them is perfect. No matter what you do, there will always be so-called <u>wolf</u> <u>intervals</u>.

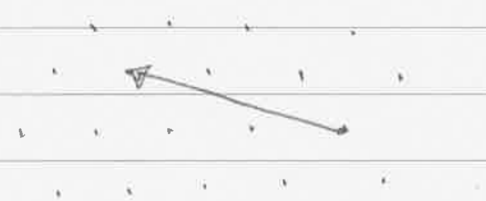
Let's look at the interval $D^{0} - A^{-1}$ from above:

$$D^0 = 9/8 \qquad A^{-1} = 5/3 \implies A^{-1}/D^0 = \frac{5/3}{9/8} = \frac{40}{27}.$$

How far is this from a perfect fifth?

$$\frac{3/2}{40/27} = 81/80 = 21.5 \text{ cents}.$$

This is an audible difference. Thus we would also like to avoid points separated by the vector



Good luck to you.

Eventually, people decided to abandon this system and allowed themselves to <u>temper</u> the scale using intervals smaller than the syntonic comma.

The began to use fractions of syntonic commas. In modern languge such tuning systems are called meantone temperament.

["Temperament" means that some intervals have been adjusted (or tempered) from their just values so that the scale can work in more keys.]

This became quite an artform. The most common scheme is known as quarter-comma meantone:

$$E^{-1} \quad B^{-5/4} \quad F_{\#}^{-3/2} \quad C_{\#}^{-7/4}$$

$$C^{0} \quad G^{-1/4} \quad D^{-1/2} \quad A^{-3/4}$$

$$Ab^{+1} \quad Eb^{+3/4} \quad Bb^{+1/2} \quad F^{+1/4}$$

However, it is by no means easy to tune a piano this way. [Note that 1/2 and 1/4 commas are irrational

$$\frac{1}{2} \text{ comma} = (81/80)^{1/2} = 9/20\sqrt{5}$$

$1/4$ comma $= 3/10 \cdot 5^{3/4}$.

At this point there seems very little reason not to go all the way to equal temperament;

and that's what people did.

THE END.