3/18/15

Today: Quiz 3 (20 minutes)

Where were we? So far we have seen:

- Introduction
  - Helmholtz' explanation of Pythagoras

- Trigonometry
  - The mathematics of beats

- Waves
  - The harmonic oscillator
  - The Wave Equation.

The next two topics are:

- Fourier Series

- Scales & Temperaments.

After that, who knows?

Recall the Wave Equation

$$u_{tt} = c^2 u_{xx}$$

D'Alembert's general solution is

$$u(x,t) = \varphi(x+ct) + \psi(x-ct)$$

where φ and ψ are arbitrary (?) functions from $\mathbb{R}$ to $\mathbb{R}$.

If we assume <u>boundary</u> conditions

$$u(0,t) = u(\ell,t) = 0 \quad \forall\, t$$

then Daniel Bernoulli says

$$u(x,t) = \sum_{n\geq 0} \sin\left(\frac{n\pi x}{\ell}\right)\left[a_n \cos\left(\frac{n\pi c t}{\ell}\right) + b_n \sin\left(\frac{n\pi c t}{\ell}\right)\right]$$

for arbitrary (?) constants $a_n, b_n \in \mathbb{R}$.

OK, fine. But how do we apply this in the real world?

In other words how can we choose the constants a_n, b_n to fit given <u>initial conditions</u>?

At time $t=0$, the string has a certain shape

$$u(x,0) = f(x)$$

and a certain velocity distribution

$$\frac{\partial u}{\partial t}(x,0) = g(x).$$

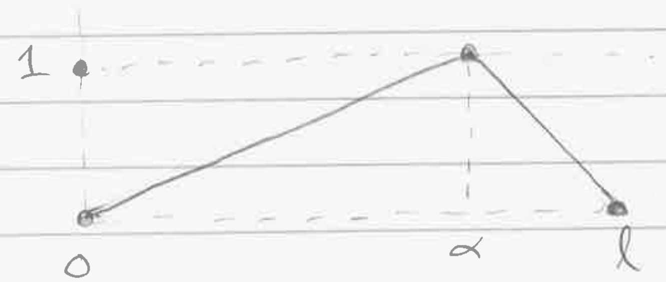
Here f and g are arbitrary (?) real valued functions defined on $[0,\ell]$.

Example: The "plucked string" has initial shape given by

$$f_\alpha(x) = \begin{cases} x/\alpha & 0 \le x \le \alpha \\ (\ell - x)/(\ell-\alpha) & \alpha \le x \le \ell \end{cases}$$

for some $\alpha \in [0,\ell]$

Picture:



To compute a_n, b_n substitute the initial conditions into Bernoulli's solution to get

$$f(x) = \sum_{n \geq 0} \sin\left(\frac{n\pi x}{\ell}\right)\left[a_n \cdot 1 + b_n \cdot 0\right]$$

$$= \sum_{n \geq 0} a_n \sin\left(\frac{n\pi x}{\ell}\right)$$

and

$$g(x) = \sum_{n \geq 0} \sin\left(\frac{n\pi x}{\ell}\right)\left[-\frac{n\pi c}{\ell} a_n \cdot 0 + \frac{n\pi c}{\ell} b_n \cdot 1\right]$$

$$= \sum_{n \geq 0} \frac{n\pi c}{\ell} b_n \sin\left(\frac{n\pi x}{\ell}\right).$$

We need to solve the following equations for a_n and b_n:

$$\boxed{\begin{aligned} f(x) &= \sum_{n \geq 0} a_n \sin\left(\frac{n\pi x}{\ell}\right) \\ g(x) &= \sum_{n \geq 0} \frac{n\pi c b_n}{\ell} \sin\left(\frac{n\pi x}{\ell}\right) \end{aligned}}$$

Is this even possible? Euler doubted it. In particular, he didn't think the plucked string could be solved this way because it has a non-differentiable cusp.

Progress was stalled until a revolutionary 1807 paper by Joseph Fourier called "Treatise on the propagation of heat in solid bodies". (Euler died in 1783.)

Fourier gave a simple method to compute the coefficients a_n, b_n (now called "Fourier coefficients").

☆ Theorem (Fourier's Trick):

Given $f(x) = \sum_{n \geq 0} a_n \sin\left(\frac{n\pi x}{\ell}\right)$ we have

$$\boxed{a_n = \frac{2}{\ell} \int_0^\ell f(x) \sin\left(\frac{n\pi x}{\ell}\right) dx}$$

Note that $f(x)$ must be nice enough so the integral exists, but it can still be pretty bad.

It remained to determine in what sense the "Fourier series"

$$f(x) = \sum_{n \geq 0} a_n \sin\left(\frac{n\pi x}{\ell}\right)$$

converges. The first theorem of this kind was given by Dirichlet (1829) and this problem led to an explosion of sophisticated analysis. But it is good to remember that Fourier's simple trick is the reason for it all.

Next time we will use Fourier's trick to create some nice animations in Maple.

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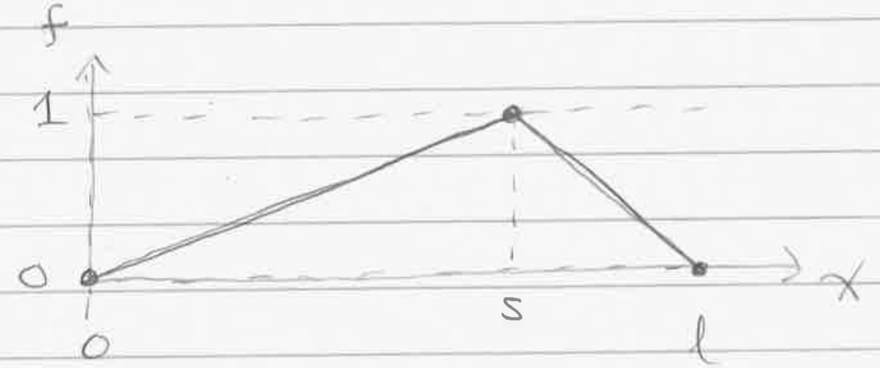
Quiz 3 was challenging.

Total	10
Average	4.94
Median	4.5
St. Dev.	3

Today we will use Fourier's Trick to create animations of waves.

Our first example is the "plucked string"



What is the equation of this curve?

For $x \in [0, s]$ it is a line of slope $1/s$ through the point $(0,0)$

$\Downarrow$

so it has equation

$$\frac{1}{s} = \frac{f-0}{x-0}$$

$$\Longrightarrow \quad f = x/s .$$

For $x \in [s, l]$ it is a line of slope $-1/(l-s)$ through the point $(l, 0)$ so it has equation

$$\frac{-1}{l-s} = \frac{f-0}{x-l}$$

$$\Longrightarrow \quad f = {}^{l-x}/_{l-s}$$

We conclude that

$$f_s(x) = \begin{cases} x/s & 0 \le x \le s \\ (l-x)/(l-s) & s \le x \le l \end{cases}$$

Now we want to solve the WE

$$u_{tt} = c^2 u_{xx}$$

with boundary conditions

$$u(0,t) = u(\ell,t) = 0 \quad \forall\ t \in \mathbb{R}.$$

and initial conditions

$$\left.\begin{aligned} u(x,0) &= f_s(x) \\ \frac{\partial u}{\partial t}(x,0) &= 0 \end{aligned}\right\} \quad \forall\ x \in [0,\ell].$$

Substituting these into Bernoulli's solution gives

$$u(x,t) = \sum_{n\geq 1} a_n \sin\left(\frac{n\pi x}{\ell}\right)\cos\left(\frac{n\pi c t}{\ell}\right)$$

where the "Fourier coefficients" a_n satisfy

$$f_s(x) = \sum_{n\geq 1} a_n \sin\left(\frac{n\pi x}{\ell}\right).$$

Fourier's Trick says that

$$a_n = \frac{2}{\ell}\int_0^\ell f_s(x)\sin\left(\frac{n\pi x}{\ell}\right)dx$$

$$= \frac{2}{\ell}\int_0^s \frac{x}{c}\sin\left(\frac{n\pi x}{\ell}\right)dx \qquad (1)$$

$$+ \frac{2}{\ell}\int_s^\ell \frac{\ell-x}{\ell-c}\sin\left(\frac{n\pi x}{\ell}\right)dx \qquad (2)$$

To compute (1) & (2) we will use the product rule

$$(uv)' = u'v + uv'$$

$$uv = \int u'v + \int uv'$$

Let $u = x$ and $v' = \sin(\omega x)$, so that $v = -\cos(\omega x)/\omega$. Then

$$\int x\sin(\omega x) = \int uv'$$

$$= uv - \int u'v$$

$$= -\frac{x\cos(\omega x)}{\omega} - \int 1\cdot\left(-\frac{\cos(\omega x)}{\omega}\right)$$

$$= -\frac{x\cos(\omega x)}{\omega} + \frac{\sin(\omega x)}{\omega^2}.$$

Then ① is

$$\frac{2}{l}\left[-x\frac{l}{n\pi}\cos\left(\frac{n\pi x}{l}\right) + \frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{l}\right)\right]_0^s$$

$$= \frac{2}{l}\left[\left(-\frac{sl}{n\pi}\cos\left(\frac{n\pi s}{l}\right) + \frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi s}{l}\right)\right) - (-0+0)\right]$$

and ② is

$$\frac{2l}{l(l-s)}\int_c^l \sin\left(\frac{n\pi x}{l}\right)dx - \frac{2}{l(l-s)}\int_c^l x\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2l}{l(l-s)}\left[-\frac{l}{n\pi}\cos\left(\frac{n\pi l}{l}\right) + \frac{l}{n\pi}\cos\left(\frac{n\pi s}{l}\right)\right]$$

$$-\frac{2}{l(l-s)}\left[\left(\frac{-l^2}{n\pi}\cos\left(\frac{n\pi l}{l}\right) + \frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi l}{l}\right)\right)\right.$$

$$\left.-\left(-\frac{sl}{n\pi}\cos\left(\frac{n\pi s}{l}\right) + \frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi s}{l}\right)\right)\right].$$

OK, fine. My computer simplifies this to

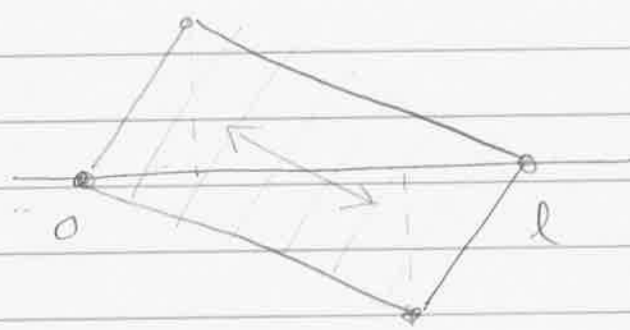
$$a_n = ① + ②$$

$$= \frac{2\ell^2}{n^2\pi^2 s(\ell - s)} \sin\left(\frac{n\pi s}{\ell}\right)$$

Therefore the solution of the plucked string is

$$u(x,t) = \sum_{n\geq 1} \frac{2\ell^2}{n^2\pi^2 s(\ell-s)} \sin\left(\frac{n\pi s}{\ell}\right) \sin\left(\frac{n\pi x}{\ell}\right) \cos\left(\frac{n\pi c t}{\ell}\right)$$

My computer can produce an animated plot of this.



[see computer]

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HW 3 & Quiz 4 TBA.

Problem: Given any reasonable function

$$f : [0, 2\pi] \longrightarrow \mathbb{R},$$

we would like to express it as a "superposition" of sine and cosine waves

(*) $$f(x) = a_0 + \sum_{n \geq 1} \left[a_n \cos(nx) + b_n \sin(nx) \right]$$

Prior to 1807, people were skeptical that this could be done. Then in 1807 Joseph Fourier came up with a trick to compute the coefficients a_0, a_n, b_n.

☆ Fourier's Trick: For $n \geq 1$ we have

$$\boxed{\begin{aligned} a_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx)\, dx \\ b_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx)\, dx. \end{aligned}}$$

The coefficient a_0 is just the average value of $f(x)$ on the interval $[0, 2\pi]$.

$$\boxed{a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx}$$

//

Last time we used F.T. to produce cool animations of vibrating string. Now I'll try to explain what's behind it.

But first, let's prove it.

Proof of F.T.:

On HW 1 Problem 5 you verified the following "orthogonality relations"

$$\frac{1}{\pi}\int_0^{2\pi} \sin(mx)\sin(nx) = \begin{cases} 1 & m=n\neq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{1}{\pi}\int_0^{2\pi} \cos(mx)\cos(nx) = \begin{cases} 2 & m=n=0 \\ 1 & m=n\neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Let me add to these one more :

$$\frac{1}{\pi} \int_0^{2\pi} \cos(mx) \sin(nx) = 0 \quad \forall\, m, n \in \mathbb{Z}.$$

This is true because the function $\cos(mx)\sin(nx)$ is <u>antisymmetric</u> about $x = \pi$, i.e.,

$$\cos(m(\pi + x)) \sin(n(\pi + x))$$

$$= -\cos(m(\pi - x)) \sin(n(\pi - x)).$$

Thus the integrals from $0 \to \pi$ and from $\pi \to 2\pi$ cancel each other. ///

Now assume that we have

$$f(x) = a_0 + \sum_{n \geq 1} \left[a_n \cos(nx) + b_n \sin(nx) \right]$$

First we comute the average value of $f(x)$ on $[0, 2\pi]$.

↓

$$\frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} a_0\,dx$$

$$+ \sum_{n\geq 1}\left[a_n \frac{1}{2\pi}\underbrace{\cancel{\int_0^{2\pi} \cos(nx)\,dx}}_{0} + b_n \frac{1}{2\pi}\underbrace{\cancel{\int_0^{2\pi} \sin(nx)\,dx}}_{0} \right]$$

$$= \frac{1}{2\pi}\Big[x a_0\Big]_0^{2\pi} = \frac{2\pi}{2\pi} a_0 = a_0 .$$

Next we integrate $f(x)$ against $\cos(mx)$

$$\frac{1}{\pi}\int_0^{2\pi} f(x)\cos(mx)\,dx = \frac{1}{\pi}\int_0^{2\pi} a_0 \underbrace{\cancel{\cos(mx)\,dx}}_{0}$$

$$+ \sum_{n\geq 1}\left[a_n \frac{1}{\pi}\underbrace{\cancel{\int_0^{2\pi} \cos(mx)\cos(nx)\,dx}}_{\substack{0 \text{ if } n \neq m \\ 1 \text{ if } n = m}} + b_n \frac{1}{\pi}\underbrace{\cancel{\int_0^{2\pi} \cos(mx)\sin(nx)\,dx}}_{0} \right]$$

$$= a_m \cdot 1 = a_m .$$

Finally we integrate $f(x)$ against $\sin(mx)$.

$$\frac{1}{\pi}\int_0^{2\pi} f(x)\sin(mx)\,dx =$$

$$\frac{1}{\pi}\int_0^{2\pi} a_0 \underbrace{\cancel{\sin(mx)}\,dx}_{0}$$

$$+ \sum_{n\geq 1}\left[a_n \frac{1}{\pi}\underbrace{\int_0^{2\pi}\cancel{\cos(nx)\sin(mx)\,dx}}_{0} + b_n \frac{1}{\pi}\int_0^{2\pi}\cancel{\sin(nx)\sin(mx)\,dx}\right]$$

0 if $n \neq m$

1 if $n = m$

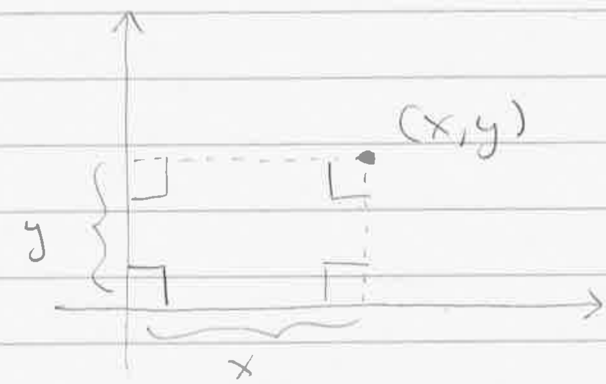
$$= b_m \cdot 1 = b_m.$$

$\blacksquare$

That was a nice trick, but what's behind it? Is there more to it than just miraculous trig identities?

To understand this we need to go back to <u>linear</u> <u>algebra</u>.

Recall Descartes' Idea of "analytic geometry".



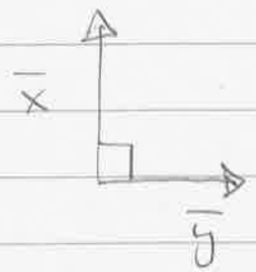
We set up two perpendicular axes and then project the point orthogonally onto each axis. That's a lot of right angles.

The modern way to talk about right angles is via the <u>dot</u> <u>product</u>.

Given $\bar{x} = (x_1, \dots, x_n)$ and $\bar{y} = (y_1, \dots, y_n)$ in "real cartesian n-space" $\mathbb{R}^n$ we define

$$\bar{x} \bullet \bar{y} := x_1 y_1 + x_2 y_2 + \cdots + x_n y_n .$$

We think of $\bar{x}$ & $\bar{y}$ as arrows and we say they are orthogonal (and write $\bar{x} \perp \bar{y}$) if the angle between them is 90°.



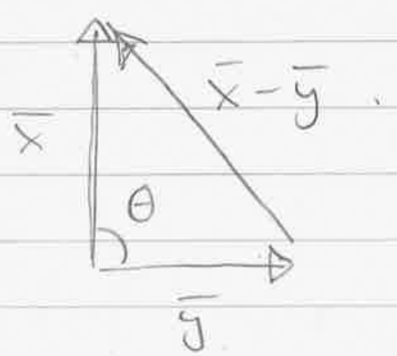
$\bar{x}$ and $\bar{y}$ live in "n-dimensional space" but together they span a "2-dimensional plane", which I know how to draw.

☆ Pythagorean Theorem:

Given nonzero $\bar{x}, \bar{y} \in \mathbb{R}^n$ we have

$$\bar{x} \perp \bar{y} \iff \bar{x} \cdot \bar{y} = 0 .$$

<u>Proof</u>: Consider the triangle of vectors with sides $\bar{x}$, $\bar{y}$, $\bar{x}-\bar{y}$, and let θ be the angle between $\bar{x}$ and $\bar{y}$.



The classical Pythagorean Theorem says that

$$\bar{x} \perp \bar{y} \iff \|\bar{x}-\bar{y}\|^2 = \|\bar{x}\|^2 + \|\bar{y}\|^2$$

What does the dot product have to do with this? We actually <u>define</u> the length of a vector by

$$\|\bar{x}\|^2 = \bar{x} \bullet \bar{x} = x_1^2 + x_2^2 + \cdots + x_n^2 .$$

Then using algebraic properties of the dot product gives

$$\begin{aligned} \|\bar{x}-\bar{y}\|^2 &= (\bar{x}-\bar{y}) \bullet (\bar{x}-\bar{y}) \\ &= \bar{x}\bullet\bar{x} - 2\bar{x}\bullet\bar{y} + \bar{y}\bullet\bar{y} \\ &= \|\bar{x}\|^2 + \|\bar{y}\|^2 - 2(\bar{x}\bullet\bar{y}) . \end{aligned}$$

We conclude that

$$\|\bar{x}-\bar{y}\|^2 = \|\bar{x}\|^2 + \|\bar{y}\|^2 \iff \bar{x}\bullet\bar{y} = 0 .$$

$\blacksquare$

This tells us that the whole system of Cartesian coordinates can be expressed in terms of the dot product.

Let $\bar{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \bar{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \cdots, \bar{e}_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^n$

be the standard basis vectors for $\mathbb{R}^n$. Every vector $\bar{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$ can be expressed uniquely in terms of the standard basis by

$$\bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + x_n \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$= x_1 \bar{e}_1 + x_2 \bar{e}_2 + \cdots + x_n \bar{e}_n .$$

The coefficients in the expansion are just the "coordinates" of $\bar{x}$.

But what if I'm given a vector $\bar{a} \in \mathbb{R}^n$ and I don't already know the coordinates? How can I compute them?

Let $\bar{a} = a_1 \bar{e}_1 + a_2 \bar{e}_2 + \cdots + a_n \bar{e}_n$

We wish to compute to coefficients a_i.

Here's the Trick: Since the basis vectors $\bar{e}_i$ are mutually orthogonal and each has length 1 we have

$$\bar{e}_i \cdot \bar{e}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Jargon: We say that the $\bar{e}_i$ form an ortho<u>nor</u>mal basis of $\mathbb{R}^n$.

Orthonormal bases are good things to have because they are really easy to work with. In particular we can compute the coordinates of any vector $\bar{a} \in \mathbb{R}^n$ using the dot product.

$$\bar{a} \cdot \bar{e}_i = (a_1 \bar{e}_1 + a_2 \bar{e}_2 + \cdots + a_n \bar{e}_n) \cdot \bar{e}_i$$

$$= a_1 \underbrace{\bar{e}_1 \cdot \bar{e}_i}_{0} + \cdots + a_i \underbrace{\bar{e}_i \cdot \bar{e}_i}_{1} + \cdots + a_n \underbrace{\bar{e}_n \cdot \bar{e}_i}_{0}$$

$$= a_i .$$

We conclude that

$$a_i = \bar{a} \cdot \bar{e}_i \quad \forall\, i = 1, 2, \ldots, n.$$

That looks just like Fourier's Trick.

3/26/15

HW 3 & Quiz 4 still TBA.

The step from 2- and 3-dimensional space to "higher-dimensional space" was a major conceptual leap. It happened some time between 1827 (Möbius' "Barycentric Calculus") and 1844 (Grassmann's "Theory of Extension").

Today we will make a conceptual leap of equal magnitude.

Definition: We define "n-dimensional Euclidean space" as the set

$$\mathbb{R}^n = \{(x_1, x_2, \cdots, x_n) : x_i \in \mathbb{R}\}$$

together with the "dot product" function

$$\bullet : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

defined by

$$\overline{x} \bullet \overline{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n .$$

This definition is meant to encapsulate the idea of "Euclidean geometry" in a way that naturally works for all dimensions.

We define the <u>length</u> of a vector by

$$\|\bar{x}\|^2 := \bar{x} \cdot \bar{x},$$

the distance between two vectors by

$$\text{dist}(\bar{x}, \bar{y}) = \|\bar{x} - \bar{y}\|$$

and the angle between two vectors by

$$\cos(\text{angle}(\bar{x}, \bar{y})) = \frac{\bar{x} \cdot \bar{y}}{\|\bar{x}\| \cdot \|\bar{y}\|}.$$

Anything "geometric" can be done in this language. //

Now we can stop worrying about what higher-dimensional space "is" and simply work with it.

"Go forward, and faith will come to you."

— D'Alembert, talking about Calculus.

Simultoneously, the revolution of "non-Euclidean geometry" made people willing to apply the word "space" to a broader range of phenomena.

So what is it about Euclidean space that singles it out among all the possible "spaces"?

Definition: A vector space (over $\mathbb{R}$) is a set V of "vectors" together with two operations

- Vector addition

$$u, v \in V \rightsquigarrow u+v \in V$$

- Scalar multiplication

$$u \in V, r \in \mathbb{R} \rightsquigarrow r \cdot u \in V$$

that satisfy the "obvious" axioms:

- $u+(v+w) = (u+v)+w \quad \forall\, u,v,w \in V$
- $u+v = v+u \quad \forall\, u,v \in V$.
- $\exists\, 0 \in V,\ \forall v \in V,\ v+0 = v$
- $v+(-1)\cdot v = 0 \quad \forall\, v \in V$
- $1\cdot v = v \quad \forall\, v \in V$
- $r\cdot(u+v) = r\cdot u + r\cdot v \quad \forall\, r \in \mathbb{R},\ u,v \in V$.
- $(r+s)\cdot u = r\cdot u + s\cdot u \quad \forall\, r,s \in \mathbb{R},\ u \in V$.

Furthermore, we say that V is on <u>inner-product</u> <u>space</u> if there exists a function

- Inner product

$$u,v \in V \rightsquigarrow \langle u,v\rangle \in \mathbb{R}$$

that satisfies the "obvious" axioms:

- $\langle u,v\rangle = \langle v,u\rangle \quad \forall\, u,v \in V$
- $\langle r\cdot u+v, w\rangle = r\langle u,w\rangle + \langle v,w\rangle \quad \forall\, r \in \mathbb{R},\ u,v,w \in V$
- $\langle v,v\rangle \geq 0 \quad \forall\, v \in V$
- $\langle v,v\rangle = 0 \implies v = 0$

"Obvious" Observation: Euclidean space $\mathbb{R}^n$ is an example of an inner-product space with the inner-product given by the dot product.

Obvious Question:

Why would anyone put up with the extreme abstraction of these definitions?

Surprising Answer:

Because there are other, more "exotic", examples of inner-product spaces. The most important example comes from the theory of Fourier Series.

Definition: Let $L^2(0, 2\pi)$ denote the set of functions $f: [0, 2\pi] \to \mathbb{R}$ satisfying

- $f(0) = f(2\pi)$
- $\int_0^{2\pi} f(x)^2 \, dx$ exists and is finite.

Note that such functions can be "added":

$$(f+g)(x) := f(x) + g(x)$$

and multiplied by scalars $r \in \mathbb{R}$:

$$(r \cdot f)(x) := r f(x)$$

One can check that these operations make $L^2(0, 2\pi)$ into a <u>vector</u> <u>space</u>.

Furthermore, there is a "natural" inner-product. For all $f, g \in L^2(0, 2\pi)$ we define

$$\langle f, g \rangle := \int_0^{2\pi} f(x) g(x) \, dx .$$

[Exercise: Check that it satisfies the axioms.]

The "square-integrable" condition guarantees that every function $f \in L^2(0, 2\pi)$ has "finite length":

$$\|f\|^2 := \langle f, f \rangle = \int_0^{2\pi} f(x)^2\,dx < \infty .$$

///

Thus we have defined a "space" of functions. But not just a space; an "inner-product space" in which we can talk about geometric concepts such as length, distance, angle, orthogonality.

We must think about this:

- A function is not a vector (i.e., a directed line segment), but a collection of functions might behave like a collection of vectors in some abstract way.

- This might be useful. It might suggest properties of functions that we wouldn't otherwise notice, and it might allow us to hack our primate talent for spacial cognition and apply it to analysis (for which primates have no special talent).

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HW3 & Quiz 4 still TBA.

Recall: Last time we distilled the essence of "Euclidean geometry" into an abstract-algebraic definition.

Definition: An inner-product space (over $\mathbb{R}$) is a set V of "vectors" together with three operations

$$+ : V \times V \longrightarrow V \qquad \text{addition}$$
$$\cdot : \mathbb{R} \times V \longrightarrow V \qquad \text{scaling}$$
$$\langle , \rangle : V \times V \longrightarrow \mathbb{R} \qquad \text{inner-product}$$

satisfying quite a few "obvious" axioms. ///

The prototype is the set $\mathbb{R}^n$ of real n-tuples together with the "dot product"

$$\langle \bar{x}, \bar{y} \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n .$$

Q: Why do we bother with the abstraction?

A: Because there exist other, more exotic, inner-product spaces, and the main example comes from Fourier series.

Definition: Let $L^2(0, 2\pi)$ denote the set of functions $f : [0, 2\pi] \to \mathbb{R}$ such that

- $f(0) = f(2\pi)$
- $\int_0^{2\pi} f(x)^2 \, dx$ exists and is finite.

We define "pointwise" addition and scalar multiplication of functions by

$$\left.\begin{array}{l} (f+g)(x) := f(x) + g(x) \\ (r \cdot f)(x) := r \cdot f(x) \end{array}\right\} \begin{array}{c} \forall f, g \in L^2(0, 2\pi) \\ \text{and} \\ \forall r \in \mathbb{R}. \end{array}$$

and we define an inner-product by

$$\langle f, g \rangle := \frac{1}{\pi} \int_0^{2\pi} f(x) g(x) \, dx \quad \forall f, g \in L^2(0, 2\pi).$$

One can check that these operations satisfy the axioms of an inner-product space over $\mathbb{R}$.

Observation: The "square integrability" condition guarantees that each function has "finite length".

$$\|f\|^2 := \langle f, f \rangle = \frac{1}{\pi}\int_0^{2\pi} f(x)^2\,dx < \infty .$$

That's good. :)

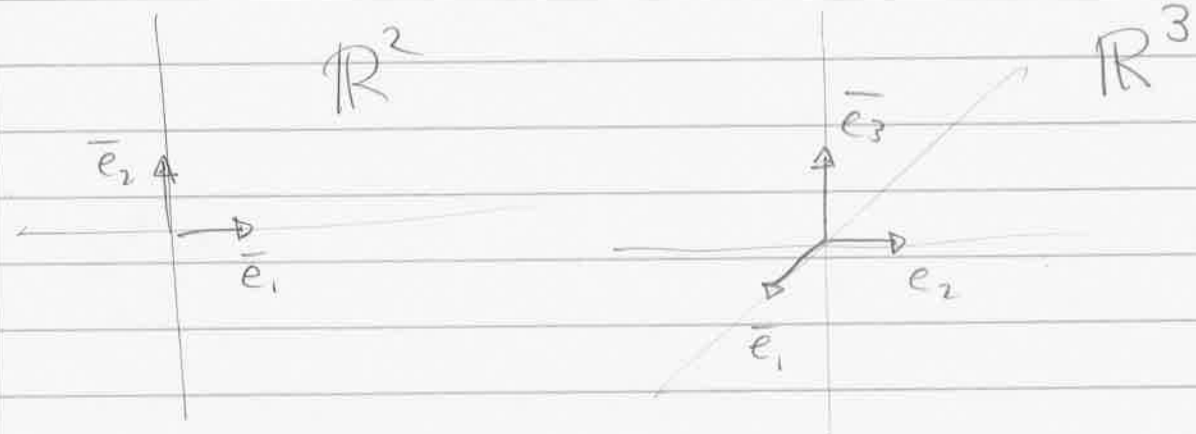
Q: How does one do computations in an inner-product space?

A: We need a "coordinate system".

In Euclidean space $\mathbb{R}^n$ there is a standard choice of coordinates, defined by the <u>standard</u> <u>basis</u> <u>vectors</u>.

$$\bar{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ \bar{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ \ldots, \ \bar{e}_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^n .$$

These are basically the Cartesian axes:



To say that $\bar{e}_1, \bar{e}_2, \dots, \bar{e}_n$ are a <u>basis</u> means that they can express each vector $\bar{a} \in \mathbb{R}^n$ in a unique way.

Example: The vector $\bar{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^n$ is expressed as

$$\bar{a} = a_1 \bar{e}_1 + a_2 \bar{e}_2 + \cdots + a_n \bar{e}_n$$

This is obvious. ///

Furthermore, we say $\bar{e}_1, \bar{e}_2, \dots, \bar{e}_n$ is an orthonormal basis because

$$\langle \bar{e}_i, \bar{e}_j \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

This allows us to compute very easily.

Given $\bar{x} \in \mathbb{R}^n$, it can be written in coordinates as

$$\bar{x} = \langle \bar{x}, \bar{e}_1 \rangle \bar{e}_1 + \langle \bar{x}, \bar{e}_2 \rangle \bar{e}_2 + \cdots + \langle \bar{x}, \bar{e}_n \rangle \bar{e}_n .$$

Proof: Exercise. ///

The <u>length</u> of $\bar{x}$ is computed by

$$\|\bar{x}\|^2 = \langle \bar{x}, \bar{x} \rangle$$

$$= \left\langle \sum_i \langle \bar{x}, \bar{e}_i \rangle \bar{e}_i , \sum_j \langle \bar{x}, \bar{e}_j \rangle \bar{e}_j \right\rangle$$

$$= \sum_{i,j} \langle \bar{x}, \bar{e}_i \rangle \langle \bar{x}, \bar{e}_j \rangle \langle \bar{e}_i, \bar{e}_j \rangle$$

$$= \sum_i \langle \bar{x}, \bar{e}_i \rangle^2 .$$

The sum of the squares of the coordinates.

NO SURPRISE.

///

OK, great.

To compute in the space $L^2(0, 2\pi)$ we need a basis of functions (preferably an orthonormal basis).

Problem: There is no obvious answer. It's not even clear whether a basis exists. Certainly there is no finite basis.

☆ Big Theorem:

The set of functions

$$\frac{1}{\sqrt{2}},\ \sin(x),\ \sin(2x),\ \sin(3x), \dots$$
$$\cos(x),\ \cos(2x),\ \cos(3x), \dots$$

are an orthonormal basis for $L^2(0, 2\pi)$. ///

The history of this theorem is tortuous. One could say it was first glimpsed in Ptolemy's theory of epicycles for describing planetary orbits, though it was never stated explicitly in that context. ("Almagest", 2nd century AD)

It was glimpsed again by Daniel Bernoulli in his solution of the Wave Equation (1755) but he stopped short of stating the theorem confidently.

The fact that these functions are orthonormal was known to Marc-Antoine Paseval in 1799. To save notation let $s_n(x) := \sin(nx)$ and $c_n(x) := \cos(nx)$. We know that for all $m, n \in \mathbb{Z}^+$ we have

$$\langle s_m(x), s_n(x) \rangle = \begin{cases} 1 & m=n \\ 0 & m \neq n \end{cases}$$

$$\langle c_m(x), c_n(x) \rangle = \begin{cases} 1 & m=n \\ 0 & m \neq n \end{cases}$$

$$\langle s_m(x), c_n(x) \rangle = 0 \quad \forall\, m, n.$$

$$\left\langle \frac{1}{\sqrt{2}}, s_n(x) \right\rangle = \left\langle \frac{1}{\sqrt{2}}, c_n(x) \right\rangle = 0 \quad \forall n$$

and, finally,

$$\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \frac{1}{\pi}\int_0^{2\pi} \left(\frac{1}{\sqrt{2}}\right)^2 dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}\, dx = 1 .$$

The fact that these functions are a basis was first claimed by Joseph Fourier in 1807. Fourier's Trick can be stated in our language as

$$f(x) = \left\langle f, \frac{1}{\sqrt{2}} \right\rangle \frac{1}{\sqrt{2}} + \sum_{n\geq 1} \Big[\langle f, c_n\rangle c_n(x) + \langle f, s_n\rangle s_n(x) \Big]$$

He gave some evidence but certainly not a full proof. The difficult issue is whether and in what sense this series of functions converges to $f(x)$.

Mathematics was not yet ready to address this issue. Lejeune Dirchlet gave the first real theorem in 1829 by proving that the series converges "pointwise" for functions $f(x)$ that are not too bad.

The search for stronger theorems led to the question: "What is a function, anyway?" from which mathematics has never recovered.

///

Corollary: To compute the "length" of a function $f \in L^2(0, 2\pi)$ we first expand it in the Fourier basis

$$f(x) = a_0 \frac{1}{\sqrt{2}} + \sum_{n \geq 1} a_n \cos(nx) + b_n \sin(nx)$$

and then use orthonormality to conclude that

$$\begin{aligned} \|f\|^2 &= \langle f, f \rangle \\ &= \frac{1}{\pi} \int_0^{2\pi} f(x)^2 \, dx \\ &= a_0^2 + \sum_{n \geq 1} (a_n^2 + b_n^2). \end{aligned}$$

This was observed by Parseval in 1799 and is often called Parseval's Theorem

$$\frac{1}{\pi}\int_0^{2\pi} f(x)^2\,dx = a_0^2 + \sum_{n\geq 1} (a_n^2 + b_n^2).$$

Q: What is the physical interpretation of this theorem in terms of the Wave Equation?

Hint: It is sometimes also called

Rayleigh's Energy Theorem.

4/2/15.

Last time we proved

Parseval's Theorem (1799):

If $f(x) = \frac{a_0}{\sqrt{2}} + \sum_{n \geq 1} \left[a_n \cos(nx) + b_n \sin(nx) \right]$

on the interval $x \in [0, 2\pi]$, then

$$\frac{1}{\pi} \int_0^{2\pi} f(x)\, dx = a_0^2 + \sum_{n \geq 1} (a_n^2 + b_n^2).$$

[Actually we gave an "algebraic" proof, ignoring issues of convergence. That's good enough for me!]

<u>Proof</u>: Think of $f \in L^2(0, 2\pi)$ and use the fact that $1/\sqrt{2}, \sin(x), \sin(2x), \ldots, \cos(x), \cos(2x), \ldots$ is an orthonormal basis to get

$$\frac{1}{\pi} \int_0^{2\pi} f(x)^2\, dx = \langle f, f \rangle$$

$$= \langle f, \tfrac{1}{\sqrt{2}} \rangle^2 + \sum_{n \geq 1} \left[\langle f, \cos(nx) \rangle^2 + \langle f, \sin(nx) \rangle^2 \right]$$

$$= a_0^2 + \sum_{n \geq 1} (a_n^2 + b_n^2).$$

Remark: This is "just" an infinite-dimensional version of the Pythagorean Theorem.

Today: What does it mean?

Let's think of $f(x)$ as the initial displacement of a stretched string:

Set up: Let $u(x,t)$ be the solution to the Wave Equation

$$u_{tt} = c^2 u_{xx}$$

with boundary conditions

$$u(0,t) = u(2\pi, t) = 0 \quad \forall t$$

and initial conditions

$$u(x,0) = f(x) \qquad \forall x \in [0, 2\pi]$$
$$u_t(x,0) = 0$$

In this situation I claim that

$$\langle f', f' \rangle = \frac{1}{\pi} \int_0^{2\pi} f'(x)^2 \, dx$$

represents the total amount of ENERGY in the string.

Q: What is "energy"?

A: That's a subtle question. Emmy Noether's Theorem (1915) says that energy is a conserved quantity arising from the <u>time symmetry</u> of the equations of mechanics.

In the case of the wave equation, we define the energy as a function of time

$$E(t) := \frac{1}{\pi} \int_0^{2\pi} \left(c^2 u_x^2(x,t) + u_t^2(x,t) \right) dx .$$

The justification for this seemingly random definition comes from the following.

☆ Theorem: Energy is conserved.

, i.e., $\frac{d}{dt} E(t) = 0$.

Proof: Assuming it is permissible to bring d/dt inside the integral, we get

$$\frac{d}{dt} E(t) = \frac{d}{dt} \frac{1}{\pi} \int_0^{2\pi} \left(c^2 u_x^2 + u_t^2 \right) dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} \left(c^2 \frac{d}{dt} u_x^2 + \frac{d}{dt} u_t^2 \right) dx .$$

The chain rule says that

$$\frac{d}{dt} u_x^2 = 2 u_x u_{xt}$$

$$\frac{d}{dt} u_t^2 = 2 u_t u_{tt}$$

so we get

$$\frac{d}{dt}E(t) = \frac{2}{\pi}\int_0^{2\pi}\left(c^2 u_x u_{xt} + u_t u_{tt}\right)dx$$

Then the fact that $u(x,t)$ is a solution to the WE (i.e., $u_{tt} = c^2 u_{xx}$) gives

$$\frac{d}{dt}E(t) = \frac{2c^2}{\pi}\int_0^{2\pi}\left(u_x u_{xt} + u_t u_{xx}\right)dx.$$

$$= \frac{2c^2}{\pi}\left[\int_0^{2\pi} u_x u_{xt}\,dx + \int_0^{2\pi} u_t u_{xx}\,dx\right]$$

To show that this is zero we will use integration by parts:

$$\int_0^{2\pi} u_x u_{xt}\,dx = \underbrace{\cancel{u_x u_t\Big|_{x=0}^{x=2\pi}}}_{0} - \int_0^{2\pi} u_{xx} u_t\,dx.$$

Note that $u_x(0,t)\,u_t(0,t) = u_x(2\pi,t)\,u_t(2\pi,t) = 0$ for all t because of the boundary conditions.

$\blacksquare$

Remark: If the boundary conditions were different this definition of energy might <u>not</u> be conserved. [Energy might leak across the boundary.]

Ok, great. Since the energy is constant we can compute its value at any time. Let's use $t=0$. Then

$$E = E(0) = \frac{1}{\pi}\int_0^{2\pi} \left(c^2 u_x^2(x,0) + u_t^2(x,0) \right) dx$$

Since $u_t(x,0) = 0$
and $u_x(x,0) = \frac{d}{dx} u(x,0) = \frac{d}{dx} f(x) = f'(x)$

we have

$$E = \frac{1}{\pi}\int_0^{2\pi} \left(c^2 f'(x)^2 + 0^2 \right) dx$$

$$= \frac{c^2}{\pi}\int_0^{2\pi} f'(x)^2 \, dx$$

$$= c^2 \langle f', f' \rangle .$$

Conclusion: Up to a constant multiple,

$$\langle f', f' \rangle = \frac{1}{\pi} \int_0^{2\pi} f'(x)^2 \, dx$$

is the total amount of energy in the string.

[For this reason we often restrict attention to solutions of finite energy $\langle f', f' \rangle < \infty$. Note that this is not necessarily implied by $f \in L^2$.]

Thus the "geometry" of L^2 space somehow encodes the "dynamics" of the wave equation.

$$\underset{\text{WE}}{\text{energy}} = \underset{L^2}{\text{inner-product}}$$

[Remark: There exist different inner products on L^2 that correspond to different kinds of differential equations. For example,

$$\langle f, g \rangle_{\text{weird}} := \int_0^{2\pi} x f(x) g(x) \, dx$$

is related to the vibrations of a circular drum head. Instead of trig functions, $\langle , \rangle_{weird}$ has an othogonal basis given by the "Bessel functions".]

Q: How much energy in a pure sine wave?

A: Let $f(x) = a \cdot \sin(nx)$, so that $f'(x) = na \cdot \cos(nx)$. Then the energy is

$$\langle f', f' \rangle = \langle na\cos(nx), na\cos(nx) \rangle$$

$$= n^2 a^2 \underbrace{\langle \cos(nx), \cos(nx) \rangle}_{1}$$

$$= n^2 a^2$$

The energy in the n^{th} fundamental mode is proportional to n^2 times the amplitude squared.

This is the correct way to think about Parseval's Theorem.

Let $f(x) = \frac{a_0}{\sqrt{2}} + \sum_{n \geq 1} a_n \cos(nx) + b_n \sin(nx)$.

so that

$$f'(x) = \sum_{n \geq 1} -na_n \sin(nx) + nb_n \cos(nx).$$

$$= \sum_{n \geq 1} n \underbrace{\sqrt{a_n^2 + b_n^2}}_{\substack{\text{amplitude} \\ \text{of the } n\text{th harmonic.}}} \sin(nx + \varphi_n)$$

Then Parseval says

$$\underbrace{\frac{1}{\pi} \int_0^{2\pi} f'(x)^2 \, dx}_{\substack{\text{Total energy} \\ \text{in the string}}} = \sum_{n \geq 1} \underbrace{n^2 (a_n^2 + b_n^2)}_{\substack{\text{Energy in the} \\ n\text{th harmonic}}}.$$

This finally explains the remarks I made on the second day of class (1/15/15).

4/7/15

Quiz 4 is on Tues Apr 14.
It will be based explicitly on the review session in today's class

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Review for Quiz 4:

The wave equation in 1D is

$$u_{tt} = c^2 u_{xx}$$

where c is the speed of propagation.
In the case of a stretched string we have

$$c = \sqrt{\frac{T}{\rho}}$$

where T is the tension and ρ is the linear density. We are interested in solutions with fixed ends. Let

$$u(0,t) = u(2\pi, t) = 0 \quad \forall t$$

"boundary conditions".

In this case, Daniel Bernoulli's solution is

$$(*) \quad u(x,t) = \sum_{n \geq 1} \sin\left(\tfrac{n}{2}x\right)\left[a_n \cos\left(\tfrac{nc}{2}t\right) + b_n \sin\left(\tfrac{nc}{2}t\right)\right]$$

where the constants a_n, b_n are called "Fourier coefficients".

To compute the coefficients we must specify an initial shape and velocity distribution

$$u(x,0) = f(x)$$
$$u_t(x,0) = g(x)$$

"initial conditions".

Putting $t=0$ in $(*)$ gives

$$f(x) = \sum_{n \geq 1} a_n \sin\left(\tfrac{n}{2}x\right).$$

Putting $t=0$ in $u_t(x,t)$ from $(*)$ gives

$$g(x) = \sum_{n \geq 1} \frac{nc}{2} b_n \sin\left(\tfrac{n}{2}x\right)$$

Then we can use "Fourier's Trick" to compute

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(2x)\sin(nx)\,dx$$

$$b_n = \frac{2}{n\pi c}\int_0^{2\pi} g(2x)\sin(nx)\,dx$$

Then we have

$$u(x,t) = \sum_{n\geq 1} \underbrace{\sqrt{a_n^2+b_n^2}}_{\substack{\text{amplitude of}\\ \text{the nth harmonic}}} \sin\left(\tfrac{n}{2}x\right)\sin\left(\tfrac{nc}{2}t + \varphi_n\right)$$

phase of the nth harmonic.

Fourier's Trick can be expressed in a more highbrow language.

Let $L^2[0,2\pi]$ be the set of functions $f: [0,2\pi]$ such that

- $f(0) = f(2\pi)$
- $\int_0^{2\pi} f(x)^2\,dx < \infty$.

This is a vector space over $\mathbb{R}$, and in fact it is an inner-product space with

$$\langle f, g \rangle := \frac{1}{\pi} \int_0^{2\pi} f(x) g(x)\, dx .$$

Using this inner-product one can show that the functions

$$\frac{1}{\sqrt{2}},\ \sin(x),\ \sin(2x), \ldots,\ \cos(x),\ \cos(2x), \ldots$$

are an ortho<u>nor</u>mal set.

Thus if, for example,

$$f(x) = \sum_{n \geq 1} a_n \sin\left(\frac{n}{2} x\right)$$

then we have

$$\langle f(2x), \sin(nx) \rangle = \left\langle \sum_{m \geq 1} a_m \sin(mx), \sin(nx) \right\rangle$$

$$= \sum_{m \geq 1} a_m \underbrace{\langle \sin(mx), \sin(nx) \rangle}_{\substack{1 \text{ if } m=n \\ 0 \text{ if } m \neq n}}$$

$$= a_n \cdot 1$$

More generally, if we can express

$$f(x) = a_0 \cdot \frac{1}{\sqrt{2}} + \sum_{n \geq 1} \left(a_n \cos(nx) + b_n \sin(nx) \right)$$

then we have

$$a_0 = \left\langle f, \frac{1}{\sqrt{2}} \right\rangle$$

$$a_n = \langle f, \cos(nx) \rangle$$

$$b_n = \langle f, \sin(nx) \rangle$$

Finally, using the Pythagorean Theorem for orthonormal sets, we get

$$\langle f, f \rangle = a_0^2 + \sum_{n \geq 1} \left(a_n^2 + b_n^2 \right)$$

This is called Parseval's Theorem.

What does it have to do with the Wave Equation?

If $u_{tt} = c^2 u_{xx}$ then we define

$$E(t) := \frac{1}{\pi} \int_0^{2\pi} \left(c^2 u_x^2 + u_t^2 \right) dx$$

Using integration by parts gives

$$\frac{d}{dt} E(t) = \frac{2c^2}{\pi} u_x u_t \Big|_{x=0}^{x=2\pi}$$

$$= \frac{2c^2}{\pi} \left[u_x(2\pi,t) u_t(2\pi,t) - u_x(0,t) u_t(0,t) \right]$$

Assuming that

$$u(0,t) = u(2\pi,t) = 0 \quad \forall t$$

we also have

$$u_t(0,t) = u_t(2\pi,t) = 0 \quad \forall t$$

and hence

$$\frac{d}{dt} E(t) = 0 \quad \text{for all } t.$$

This means that "energy is conserved".

Since the energy is constant we can compute it at a convenient time, say $t=0$. Then

$$E = E(0) = \frac{1}{\pi}\int_0^{2\pi} \left(c^2 u_x^2(x,0) + u_t^2(x,0)\right) dx .$$

Now assuming that

$$u(x,0) = f(x)$$
$$u_t(x,0) = 0$$

we get $u_x(x,0) = f'(x)$ and hence

$$E = \frac{1}{\pi}\int_0^{2\pi} \left(c^2 f'(x)^2 + 0^2\right) dx$$

$$= c^2 \frac{1}{\pi}\int_0^{2\pi} f'(x)^2\, dx$$

$$= c^2 \langle f', f' \rangle .$$

The total energy in the wave is c^2 times the "L^2-length" of the slope of the initial shape.

More generally, if

$$u(x,0) = f(x)$$
$$u_t(x,0) = g(x)$$

then we have

$$E = E(0) = \frac{1}{\pi}\int_0^{2\pi}\left(c^2 f'(x)^2 + g(x)^2\right) dx$$

$$= c^2 \frac{1}{\pi}\int_0^{2\pi} f'(x)^2 \, dx + \frac{1}{\pi}\int_0^{2\pi} g(x)^2 \, dx$$

$$= \underbrace{c^2 \langle f', f' \rangle}_{\text{initial potential energy}} + \underbrace{\langle g, g \rangle}_{\text{initial kinetic energy}}$$

What does Parseval's Theorem say?

It tells us that the total energy in the wave is the sum of the energies in its harmonic components.

For example, suppose we have a wave with initial shape

$$f(x) = a_0 \frac{1}{\sqrt{2}} + \sum_{n \geq 1} \left(a_n \cos(nx) + b_n \sin(nx) \right)$$

and zero initial velocity. Then the total energy in the wave is

$$c^2 \langle f', f' \rangle .$$

On the other hand, the total energy in $a_n \cos(nx)$ is

$$c^2 \langle -n a_n \sin(nx), -n a_n \sin(nx) \rangle$$

$$= c^2 n^2 a_n^2 \underbrace{\langle \sin(nx), \sin(nx) \rangle}_{1}.$$

$$= c^2 n^2 a_n^2$$

and similarly, the total energy in $b_n \sin(nx)$ is

$$c^2 n^2 b_n^2 .$$

Finally we have

$$f'(x) = \sum_{n\geq 1} \left(-na_n \sin(nx) + nb_n \cos(nx)\right)$$

and so Parseval says

$$\langle f', f' \rangle = \sum_{n\geq 1} \left((-na_n)^2 + (nb_n)^2\right)$$

$$= \sum_{n\geq 1} \left(n^2 a_n^2 + n^2 b_n\right).$$

Multiply both sides by c^2 to get

$$\underbrace{c^2 \langle f', f' \rangle}_{\text{Total energy in the wave}} = \sum_{n\geq 1} \Big(\underbrace{c^2 n^2 a_n^2 + c^2 n^2 b_n^2}_{\text{Energy in the nth harmonic.}} \Big)$$