**Math 592 (The Math of Music)** **Spring 2015**
**Quiz 4** **Drew Armstrong**

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**1.** In the one-dimensional wave equation $u_{tt} = c^2 u_{xx}$, what does the number $c$ represent? How can you compute $c$ in the case of a stretched string?

The number $c$ is the "speed of propagation". In the case of a stretched string with tension $T$ and linear density $\rho$ we have

$$c = \sqrt{\frac{T}{\rho}}$$

**2.** Explicitly verify that $u(x,t) = \sin(x)\sin(ct)$ is a solution to $u_{tt} = c^2 u_{xx}$.

Note that

$$u_{xx}(x,t) = \frac{\partial^2}{\partial x^2}\sin(x)\sin(ct) = \frac{\partial}{\partial x}\cos(x)\sin(ct) = -\sin(x)\sin(ct)$$

and

$$u_{tt}(x,t) = \frac{\partial^2}{\partial t^2}\sin(x)\sin(ct) = \frac{\partial}{\partial t} c\sin(x)\cos(ct) = -c^2\sin(x)\sin(ct) = c^2 u_{xx}(x,t).$$

**3.** Suppose that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are orthonormal vectors in some inner product space. In that case, what is the length of the vector $\mathbf{v} = a\mathbf{u}_1 + b\mathbf{u}_2 + c\mathbf{u}_3$?

We have

$$\begin{aligned}
\|\mathbf{v}\|^2 &= \langle a\mathbf{u}_1 + b\mathbf{u}_2 + c\mathbf{u}_3, a\mathbf{u}_1 + b\mathbf{u}_2 + c\mathbf{u}_3\rangle \\
&= \begin{array}{lll} a^2\langle \mathbf{u}_1, \mathbf{u}_1\rangle & +ab\langle \mathbf{u}_1, \mathbf{u}_2\rangle & +ac\langle \mathbf{u}_1, \mathbf{u}_3\rangle \\ +ba\langle \mathbf{u}_2, \mathbf{u}_1\rangle & +b^2\langle \mathbf{u}_2, \mathbf{u}_2\rangle & +bc\langle \mathbf{u}_2, \mathbf{u}_3\rangle \\ +ac\langle \mathbf{u}_3, \mathbf{u}_1\rangle & +bc\langle \mathbf{u}_3, \mathbf{u}_2\rangle & +c^2\langle \mathbf{u}_3, \mathbf{u}_3\rangle \end{array} \\
&= \begin{array}{lll} 1a^2 & +0ab & +0ac \\ +0ba & +1b^2 & +0bc \\ +0ac & +0bc & +1c^2 \end{array} \\
&= a^2 + b^2 + c^2
\end{aligned}$$

**4.** State the definition of the inner product of functions that makes

$$1/\sqrt{2}, \sin(x), \cos(x), \sin(2x), \cos(2x), \ldots$$

into an orthonormal set.

Answer:

$$\langle f, g \rangle = \frac{1}{\pi} \int_0^{2\pi} f(x)g(x)\, dx$$

**5.** Compute all of the coefficients in the Fourier expansion of $\cos^2 x$:

$$\cos^2 x = a_0 \frac{1}{\sqrt{2}} + \sum_{n \geq 1} (a_n \cos(nx) + b_n \sin(nx))$$

[Hint: No integration is necessary. Use $\cos(2x) = \cos^2 x - \sin^2 x$.]

We have

$$\cos(2x) = \cos^2 x - \sin^2 x = \cos^2 x - (1 - \cos^2 x) = 2\cos^2 x - 1$$

and hence

$$\cos^2 x = \frac{\cos(2x) + 1}{2} = \frac{1}{2} + \frac{1}{2}\cos(2x).$$

We conclude that

$$a_0 = 1/\sqrt{2},$$
$$a_2 = 1/2,$$

and the rest of the Fourier coefficients are zero.