Math 592 **Spring 2015**
Homework 3 **Drew Armstrong**

A Justly Tuned Chromatic Scale.

On this homework you will compute the best rational approximations to the notes of the equal-tempered chromatic scale. The tool you will use is called continued fractions. Any irrational number α can be expressed uniquely in the form

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}.$$

where a_0 is an integer and $a_1, a_2, \ldots$ are positive integers. This is called the continued fraction expansion of α. To save space we will use the notation

$$\alpha = [a_0; a_1, a_2, a_3, \ldots].$$

Given integers p, q we will say that p/q is a best rational approximation of α if the quantity $|p/q - \alpha|$ is minimized among all fractions with denominators less than or equal to q. It is a theorem that the sequence of best rational approximations of α can be read off from the continued fraction expansion, as follows:

- Truncate the continued fraction at the nth place to get $[a_0; a_1, a_2, \ldots, a_n]$,
- Replace a_n by any integer between $a_n/2$ and a_n.

For example, consider the number $\pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, \ldots]$. The sequence of best rational approximations is given by

$$\begin{array}{ccccccccc} [3] & [3;4] & [3;5] & [3;6] & [3;7] & [3;7,8] & [3;7,9] & [3;7,10] & \cdots \\ 3 & \frac{13}{4} & \frac{16}{5} & \frac{19}{6} & \frac{22}{7} & \frac{179}{57} & \frac{201}{64} & \frac{223}{71} & \cdots \end{array}$$

Problem.

(a) Use this method to compute the best rational approximations to the notes of the equal-tempered chromatic scale. That is, for each value $(2^{1/12})^k$ for k from 1 to 12, choose the first fraction in the sequence of best rational approximations that is within 50 cents (half of a semitone) of the correct value. Call it r_k. [Hint: Use WolframAlpha or some other thing to compute the continued fraction expansions.]

(b) Compute the ratios between all pairs of notes separated by seven semitones. That is, compute the ratios

$$\frac{r_8}{r_1}, \frac{r_9}{r_2}, \ldots, \frac{r_{12}}{r_5}, \frac{2r_1}{r_6}, \frac{2r_2}{r_7}, \ldots, \frac{2r_7}{r_{12}}.$$

Which intervals are close to a perfect fifth and which are far away? The bad ones are called **wolf intervals** because they sound like a wolf howling. This is the main weakness of just intonation.

Many tuning systems try to tame the wolf, but it is an impossible problem. There is really no way to have small integer ratios in one key without creating wolf intervals in other keys.