Math 592 **Spring 2015**
Homework 1 **Drew Armstrong**

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0. Compute the length of a chord of the unit circle subtended by an arc of length t.

1. Given an arbitrary matrix $A = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$ we can define a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ by $\mathbf{x} \mapsto A\mathbf{x}$, in other words,

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + cy \\ bx + dy \end{pmatrix}.$$

Prove that this is a linear function.

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear function and consider the standard basis of $\mathbb{R}^2$ consisting of $\mathbf{e}_1 = (1,0)$ and $\mathbf{e}_2 = (0,1)$. If $f(\mathbf{e}_1) = (a,b)$ and $f(\mathbf{e}_2) = (c,d)$ then we define the matrix

$$[f] = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Given $\mathbf{x} \in \mathbb{R}^2$ we will write $[\mathbf{x}]$ for the corresponding column vector. Then we define the product of a matrix and a column by $[f][\mathbf{x}] = [f(\mathbf{x})]$. [Why do we do this?]

2. Let f and g be linear functions from $\mathbb{R}^2$ to $\mathbb{R}^2$.

(a) Prove that the composite $f \circ g : \mathbb{R}^2 \to \mathbb{R}^2$ is also linear.

(b) We define the matrix product by $[f][g] := [f \circ g]$. If $[f] = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$ and $[g] = \left(\begin{smallmatrix} a' & c' \\ b' & d' \end{smallmatrix}\right)$, use the definition to compute the matrix product $[f][g]$.

3. Let $R_t : \mathbb{R}^2 \to \mathbb{R}^2$ be the (linear) function that rotates the plane counterclockwise by angle t. Recall that we can express this in coordinates by

$$[R_t] = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

(a) Explain why $[R_t]^3 = [R_{3t}]$ without doing any work.

(b) Use part (a) to express $\cos(3t)$ as a polynomial in $\cos(t)$. This is an example of a Chebyshev polynomial of the first kind.

4. Use the "angle sum fomulas" to verify the following trigonometric identities.

(a) $2 \sin\alpha \sin\beta = \cos(\alpha - \beta) - \cos(\alpha + \beta)$

(b) $2 \cos\alpha \cos\beta = \cos(\alpha - \beta) + \cos(\alpha + \beta)$

5. Use the identities from Problem 4 to verify the following integrals.

(a) $\displaystyle\int_0^{2\pi} \sin(mt)\sin(nt)\,dt = \begin{cases} \pi & m = n \neq 0 \\ 0 & \text{otherwise} \end{cases}$

(b) $\displaystyle\int_0^{2\pi} \cos(mt)\cos(nt)\,dt = \begin{cases} 2\pi & m = n = 0 \\ \pi & m = n \neq 0 \\ 0 & \text{otherwise} \end{cases}$