Math 562/662 **Spring 2022**
Homework 2 **Drew Armstrong**

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1. One Step Subgroup Test. Let $(G, *, \varepsilon)$ be a group and let $H \subseteq G$ be a subset. We say that H is a *subgroup* when the following three conditions are satisfied:

- $\varepsilon \in H$,
- $a \in H \Rightarrow a^{-1} \in H$,
- $a, b \in H \Rightarrow a * b \in H$.

Prove that these three conditions are equivalent to the following single condition:

$$a, b \in H \quad \Rightarrow \quad a^{-1} * b \in H.$$

2. Congruence Modulo a Subgroup. Let $(G, *, \varepsilon)$ be a group and let $H \subseteq G$ be a subgroup. For any $a, b \in G$ we define the relation of *congruence modulo H*:

$$a \equiv b \bmod H \quad \Longleftrightarrow \quad a^{-1} * b \in H.$$

And for any $a \in G$ we define the *coset of H generated by a*:

$$a * H := \{a * h : h \in H\} \subseteq G.$$

(a) Prove that congruence mod H is an equivalence relation on G.
(b) For all $a, b \in G$, prove that a and b are congruent mod H if and only if the cosets that they generate are equal:

$$a \equiv b \bmod H \quad \Longleftrightarrow \quad a * H = b * H.$$

3. Orbit-Stabilizer Theorem. Let $(G, *, \varepsilon)$ be a group and let X be a set. Consider a function $\cdot : G \times X \to X$, which we will denote by $(g, x) \mapsto g \cdot x$. We call this function an *action of G on X* when the following two properties are satisfied:

- $\varepsilon \cdot x = x$ for all $x \in X$,
- $a \cdot (b \cdot x) = (a * b) \cdot x$ for all $a, b \in G$ and $x \in X$.

(a) For any element $x \in X$ we define the set $\text{Stab}(x) := \{a \in G : a \cdot x = x\} \subseteq G$, called the *stabilizer of x*. Prove that this set is a subgroup of G.
(b) For any element $x \in X$ we define the set $\text{Orb}(x) := \{g \cdot x : g \in G\} \subseteq X$, called the *orbit of x*. Prove that there exists a bijection $\text{Orb}(x) \leftrightarrow G/\text{Stab}(x)$ between elements of the orbit and cosets of the stabilizer. [Hint: Send the element $g \cdot x \in \text{Orb}(x)$ to the coset $g * \text{Stab}(x)$. Check that this is well-defined and bijective.]
(c) If G is finite, combine (b) with Lagrange's Theorem to prove that

$$\#G = \#\text{Orb}(x)\#\text{Stab}(x) \quad \text{for any } x \in X.$$

4. The Alternating Group, Part 2. Consider the following polynomial in n variables:

$$\delta(x_1, \ldots, x_n) = \prod_{1 \leq i < j \leq n} (x_i - x_j) \in \mathbb{Q}[x_1, \ldots, x_n].$$

Recall that the symmetric group S_n acts on the ring of polynomials by permuting variables: For all $\sigma \in S_n$ and $f(x_1, \ldots, x_n) \in \mathbb{Q}[x_1, \ldots, x_n]$ we define

$$(\sigma \cdot f)(x_1, \ldots, x_n) := f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \in \mathbb{Q}[x].$$

(a) Prove that the stabilizer of δ under this action is the alternating group:

$$\text{Stab}(\delta) = A_n.$$

[Hint: Show that for any transposition $t \in S_n$ we have $t \cdot \delta = -\delta$.]

(b) Now use the Orbit-Stabilizer Theorem to prove that

$$\#A_n = \frac{1}{2}\#S_n = \frac{1}{2}n!.$$

[Hint: Show that $\text{Orb}(\delta)$ has size 2.]