**Math 562** **Spring 2014**
**Homework 3** **Drew Armstrong**

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Let $R$ be a ring. We say that $R$ is a domain if for all $a, b \in R$ we have

$$ab = 0 \quad \Longrightarrow \quad a = 0 \text{ or } b = 0,$$

that is, if the ring has no zerodivisors.

**1. Prime Ideals.** Given an ideal $I \leq R$ in a general ring $R$ we say that $I$ is prime if for all $a, b \in R$ we have

$$ab \in I \quad \Longrightarrow \quad a \in I \text{ or } b \in I.$$

(a) If $I \leq R$ is a prime ideal, prove that $R/I$ is a domain.
(b) If $R/I$ is a domain, prove that $I$ is a prime ideal.
(c) Prove that every maximal ideal is prime. [Hint: Every field is a domain.]

**2. Domain = Subring of a Field.** In this problem you will prove that $R$ is a domain if and only if $R$ is a subring of a field.

(a) If $R$ is a subring of a field $K$, prove that $R$ is a domain.
(b) Let $R$ be a domain and define the set of fractions:

$$\text{Frac}(R) := \left\{ \left[\frac{a}{b}\right] : a, b \in R, b \neq 0 \right\}.$$

At first these are just abstract symbols. We define a relation on $\text{Frac}(R)$ by saying that $\left[\frac{a}{b}\right] = \left[\frac{c}{d}\right]$ if and only if $ad = bc$. Prove that this is an **equivalence relation**.

(c) Now we define "multiplication" and "addition" of fractions by:

$$\left[\frac{a}{b}\right] \cdot \left[\frac{c}{d}\right] := \left[\frac{ac}{bd}\right] \quad \text{and} \quad \left[\frac{a}{b}\right] + \left[\frac{c}{d}\right] := \left[\frac{ad + bc}{bd}\right].$$

Prove that these operations are **well-defined**.

(d) It follows that $\text{Frac}(R)$ is a field (you don't need to check this) since for all nonzero $\left[\frac{a}{b}\right]$ we have $\left[\frac{a}{b}\right]^{-1} = \left[\frac{b}{a}\right]$. Prove that the map $\iota : R \to \text{Frac}(R)$ defined by $\iota(a) := \left[\frac{a}{1}\right]$ is an injective ring homomorphism. Use the First Isomorphism Theorem to conclude that $R$ is isomorphic to a subring of its field of fractions $\text{Frac}(R)$.

**3. Prime $\Longrightarrow$ Maximal in a PID.** In Problem 1 we saw that every maximal ideal in a general ring is prime. Now let $R$ be a PID. We will see that every prime ideal in $R$ is maximal.

(a) Let $I \leq R$ be a prime ideal. Since $R$ is a PID we have $I = (p)$ for some $p \in R$. Show that for all $a, b \in R$ we have

$$p | ab \quad \Longrightarrow \quad p | a \text{ or } p | b.$$

We say that $p \in R$ is a prime element.

(b) We say that $a \in R$ is an irreducible element if for all $b, c \in R$ we have

$$a = bc \quad \Longrightarrow \quad b \text{ or } c \text{ is a unit.}$$

Prove that every irreducible element in a PID is irreducible.

(c) Use this to conclude that every prime ideal in a PID is maximal. [Hint: Let $I \leq R$ be a prime ideal. Then $I = (p)$ for some prime element $p \in R$. By part (c), this $p$ is also irreducible. Then what?]

**4. Polynomials Over a Domain.** Let $R$ be a domain and consider the ring $R[x]$. Given a polynomial $f(x) = \sum_{k \geq 0} a_k x^k \in R[x]$ we define $\deg(f)$ to be the largest $k$ such that $a_k \neq 0$.

(a) Given $f, g \in R[x]$ prove that $\deg(fg) = \deg(f) + \deg(g)$.
(b) Prove that $R[x]$ is a domain.
(c) Prove that the group of units is $R[x]^\times = R^\times$.
(d) Give a specific example to show that (c) can fail when $R$ is not a domain. [Hint: Let $R = \mathbb{Z}/4\mathbb{Z}$. Show that the polynomial $1 + 2x \in (\mathbb{Z}/4\mathbb{Z})[x]$ is a unit.]

**5. Prime $\not\Rightarrow$ Maximal in General.**

(a) Let $I \leq R$ be an ideal in a general ring and consider the map

$$\varphi : R[x] \to (R/I)[x]$$

defined by $\sum_k a_k x^k \mapsto \sum_k (a_k + I) x^k$. Show that $\varphi$ is a surjective ring homomorphism.

(b) Show that the kernel of $\varphi$ is the set

$$I[x] := \left\{ \sum_k a_k x^k \in R[x] : a_k \in I \text{ for all } k \right\},$$

and hence $I[x] \leq R[x]$ is an ideal.

(c) Use the First Isomorphism Theorem to conclude that $(R/I)[x] \approx (R[x])/(I[x])$.

(d) Consider the prime (hence maximal) ideal $3\mathbb{Z}$ in the PID $\mathbb{Z}$. Show that $3\mathbb{Z}[x]$ is a prime ideal of $\mathbb{Z}[x]$ that is not maximal. Conclude that $\mathbb{Z}[x]$ is not a PID. [Hint: Use Problem 4 to show that $(\mathbb{Z}/3\mathbb{Z})[x]$ is a domain but not a field. Use part (c) and Problem 1 to conclude that $3\mathbb{Z}[x]$ is prime but not maximal. Use Problem 3 to conclude that $\mathbb{Z}[x]$ is not a PID.]