Math 562 **Spring 2012**
Homework 4 **Drew Armstrong**

Problems on Rings

1. We say that an ideal $I \subseteq R$ is prime if for all $a, b \in R$, $ab \in I$ implies that $a \in I$ or $b \in I$.

(a) Prove that $I \subseteq R$ is prime if and only if R/I is an integral domain.
(b) Prove that every maximal ideal is prime.

2. The following two proofs are **wrong**. Explain why, and **fix them**.

(a) Let R be an integral domain and consider a principal ideal $(a) \subseteq R$. If a is irreducible, then the ideal (a) is maximal, hence the ideal (a) is prime, hence the element a is prime. We conclude that every irreducible element is prime.
(b) Let $I \subseteq R$ be an ideal in an integral domain. If I is a prime ideal, then $I = (p)$ for some prime element $p \in R$. But every prime element of a domain is irreducible, hence p is irreducible and the ideal $I = (p)$ is maximal. We conclude that every prime ideal is maximal.

3. Given a ring R, there exists a unique ring homomorphism $\varphi : \mathbb{Z} \to R$ defined by $\varphi(1_{\mathbb{Z}}) = 1_R$. If $\ker \varphi = (n) \subseteq \mathbb{Z}$, we say the ring R has "characteristic n".

(a) Prove that the characteristic of an integral domain is 0 or prime $p \in \mathbb{Z}$.
(b) Prove that a field F has characteristic 0 if and only if it contains a subfield isomorphic to $\mathbb{Q}$.

Problems on Fields

4. Finite Implies Algebraic. Consider a field extension $F \subseteq K$. We say that $a \in K$ is algebraic over F if $f(a) = 0$ for some (monic) polynomial $f(x) \in F[x]$. We say that the extension $F \subseteq K$ is algebraic if every element of K is algebraic over F. Prove that if $[K : F] < \infty$ then $F \subseteq K$ is algebraic. [Hint: Consider the powers $1, \alpha, \alpha^2, \ldots$ of some $\alpha \in K$. Are they independent over F?]

5. Given a field extension $F \subseteq K$, let $F \subseteq \overline{F} \subseteq K$ denote the subset of elements that are algebraic over F. This is called the algebraic closure of F in K. Prove that $\overline{F}$ is a field. [Hint: Consider $a, b \in \overline{F}$ and note that $F(a, b) \subseteq K$ contains $a + b, a - b, ab$ and a/b $(= ab^{-1})$. By Problem 4, it suffices to show that $[F(a, b) : F] < \infty$.]

Problems on Galois Theory

6. Give a short proof that $\sqrt{2}$ is an element of the field $\mathbb{Q}(\sqrt{2} + \sqrt{3}) \subseteq \mathbb{R}$. [Hint: By definition, the real inverse of $\sqrt{2} + \sqrt{3}$ is also in $\mathbb{Q}(\sqrt{2} + \sqrt{3})$. What is this inverse?]

7. Let $\mathbb{Q} \subseteq K \subseteq \mathbb{C}$ be the splitting field of $x^4 + 1 \in \mathbb{Q}[x]$.

(a) Prove that $K = \mathbb{Q}(\sqrt{2}, i)$.
(b) Prove that $[K : \mathbb{Q}] = 4$ and hence the Galois group $\mathsf{Gal}(K/\mathbb{Q})$ has order 4.
(c) Prove that $\mathsf{Gal}(K/F) \approx V := \mathbb{Z}/(2) \times \mathbb{Z}/(2)$, the "Klein Viergruppe".
(d) Draw and label the lattice of fields between $\mathbb{Q}$ and K.