Math 562 **Spring 2012**
Wed Mar 28: Exam 2 **Drew Armstrong**

There are 3 problems and 4 pages. This is a closed book test. Any student caught cheating will receive a score of zero.

1. Let R be a commutative ring with 1 and let $I \subseteq R$ be a **maximal ideal**.

(a) Given $a \in R$ with $a \notin I$, **prove that** $I < (a) + I$ (i.e. strict containment of ideals).

Proof. Note that $a \in (a) + I$ but $a \notin I$. Since $I \subseteq (a) + I$ by definition, we conclude that $I < (a) + I$. □

(b) Use part (a) to **prove that there exist** $b \in R$ and $u \in I$ such that $1 = ab + u$.

Proof. Since I is maximal and $I < (a) + I$ we conclude that $(a) + I = R$. Then since $1 \in R = (a) + I$, there exist $b \in R$ and $u \in I$ such that $1 = ab + u$. □

(c) **Prove that** R/I **is a field**. (Write out the full proof in nice words, using ideas from parts (a) and (b).)

Proof. Let $a + I$ be any nonzero element of R/I (i.e. $a + I \neq I$). In particular we have $a \notin I$ so by parts (a) and (b) there exist $b \in R$ and $u \in I$ such that $1 = ab + u$. This implies that $(a + I)(b + I) = ab + I = 1 - u + I = 1 + I$ and so $a + I$ is invertible. We conclude that R/I is a field. □

2. Consider a field extension $F \subseteq K$ with $\alpha \in K$ algebraic over F. Let $(m_\alpha(x))$ be the kernel of the evaluation map $\varphi_\alpha : F[x] \to K$, where $m_\alpha(x) \in F[x]$ is monic.

(a) Prove that $m_\alpha(x)$ is irreducible.

Proof. Suppose that $m_\alpha(x) = f(x)g(x) \in F[x]$. Then applying φ_α shows that $0 = m_\alpha(\alpha) = f(\alpha)g(\alpha)$. Since F is a domain we have (without loss of generality) that $f(\alpha) = 0$ and hence $f(x) \in \ker \varphi_\alpha = (m_\alpha(x))$. Since $f(x)$ and $m_\alpha(x)$ divide each other, they are associates in $F[x]$. Hence $m_\alpha(x)$ has no proper factorization. □

(b) Using part (a), **prove that** $(m_\alpha(x)) \subseteq F[x]$ **is a maximal ideal** (hence $F[x]/(m_\alpha(x))$ is a field by Problem 1).

Proof. Suppose that there exists an ideal $J \subseteq F[x]$ with $(m_\alpha(x) < J < F[x]$. Since $F[x]$ is a PID we have $J = (f(x))$ for some $f(x) \in F[x]$, and then $f(x)$ is a proper factor of $m_\alpha(x)$. This contradicts the fact that $m_\alpha(x)$ is irreducible. Hence ideal $(m_\alpha(x)) \subseteq F[x]$ is maximal and $F[x]/(m_\alpha(x))$ is a field. □

(c) **Prove that the map** $\phi : F \to F[x]/(m_\alpha(x))$ **defined by** $\phi(a) := a + (m_\alpha(x))$ **is injective.** (Since ϕ is obviously a ring homomorphism — don't prove this — we can say that $F \approx \phi(F) \subseteq F[x]/(m_\alpha(x))$ is a field extension.)

Proof. Suppose that $\phi(a) = \phi(b)$, i.e. $a + (m_\alpha(x)) = b + (m_\alpha(x))$. Then $a - b \in (m_\alpha(x))$ implies that $m_\alpha(x)$ divides $a - b$ in $F[x]$. If $a - b \neq 0$ this means that $\deg(a - b) \geq \deg(m_\alpha(x)) \geq 1$, which is a contradiction. Hence $a - b = 0$. □

(d) The "first isomorphism theorem" turns the evaluation map $\varphi_\alpha : F[x] \to K$ into an isomorphism of fields $F[x]/(m_\alpha(x)) \xrightarrow{\sim} \operatorname{im}\varphi_\alpha =: F(a) \subseteq K$. **What is the definition of the isomorphism?**

$$\boxed{f(x) + (m_\alpha(x)) \mapsto \varphi_\alpha(f(x)) = f(\alpha)}$$

(e) Under the isomorphism, **which element of** $F[x]/(m_\alpha(x))$ **gets sent to** $\alpha \in K$?

$$\boxed{x + (m_\alpha(x)) \mapsto \alpha}$$

3. Let $\gamma = \sqrt[3]{2} \in \mathbb{R}$ and let $\omega = e^{2\pi i/3} \in \mathbb{C}$.

(a) What is the minimal polynomial of γ over $\mathbb{Q}$? (Just state it — no need for proof.)

$$\boxed{x^3 - 2 \in \mathbb{Q}[x]}$$

(b) What is the minimal polynomial of ω over $\mathbb{Q}(\gamma)$? (Again, don't prove it.)

$$\boxed{x^2 + x + 1 \in \mathbb{Q}(\gamma)[x]}$$

(c) Tell me a basis for the vector space $\mathbb{Q}(\gamma, \omega)$ over $\mathbb{Q}$. [Hint: Tower Law.]

Proof. Since $1, \gamma, \gamma^2$ is a basis for $\mathbb{Q}(\gamma)$ over $\mathbb{Q}$, and $1, \omega$ is a basis for $\mathbb{Q}(\gamma, \omega)$ over $\mathbb{Q}(\gamma)$, the Tower Law says that $1, \gamma, \gamma^2, \omega, \omega\gamma, \omega\gamma^2$ is a basis for $\mathbb{Q}(\gamma, \omega)$ over $\mathbb{Q}$. □

(d) The number $\gamma + \omega \in \mathbb{C}$ is a root of the polynomial

$$x^6 + 3x^5 + 6x^4 + 3x^3 + 9x + 9 \in \mathbb{Q}[x].$$

Furthermore, this polynomial is irreducible over $\mathbb{Q}$. (Just believe me.) Use this information to **prove that** $\mathbb{Q}(\gamma, \omega) = \mathbb{Q}(\gamma + \omega)$.

Proof. Since $\gamma + \omega \in \mathbb{Q}(\gamma, \omega)$ we have $\mathbb{Q}(\gamma + \omega) \subseteq \mathbb{Q}(\gamma + \omega)$, and the Tower Law says that

$$[\mathbb{Q}(\gamma, \omega) : \mathbb{Q}] = [\mathbb{Q}(\gamma, \omega) : \mathbb{Q}(\gamma + \omega)] \cdot [\mathbb{Q}(\gamma + \omega) : \mathbb{Q}].$$

From part (c), we know that $[\mathbb{Q}(\gamma, \omega) : \mathbb{Q}] = 6$ since $\mathbb{Q}(\gamma, \omega)$ has a basis of size 6 over $\mathbb{Q}$. We also know that $f(x) = x^6 + 3x^5 + 6x^4 + 3x^3 + 9x + 9$ is divisible by the minimal polynomial of $\gamma + \omega$ over $\mathbb{Q}$. Since $f(x)$ is irreducible and monic, this implies that $f(x)$ *is* the minimal polynomial of $\gamma + \omega$ over $\mathbb{Q}$, and hence $[\mathbb{Q}(\gamma + \omega) : \mathbb{Q}] = \deg(f) = 6$. We conclude that $[\mathbb{Q}(\gamma, \omega) : \mathbb{Q}(\gamma + \omega)] = 1$, hence $\mathbb{Q}(\gamma, \omega) = \mathbb{Q}(\gamma + \omega)$. □