Math 561/661 **Fall 2023**
Homework 1 **Drew Armstrong**

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1. Group Axioms. Let G be a set with a binary operation $(a,b) \mapsto a * b$. Consider the following four possible axioms:

(G1) For all $a,b,c \in G$ we have $a*(b*c) = (a*b)*c$.
(G2) There exists some $\varepsilon \in G$ such that $a * \varepsilon = \varepsilon * a = a$ for all $a \in G$.
(G3) For each $a \in G$ there exists some $b \in G$ such that $a * b = b * a = \varepsilon$.
(G4) For each $a \in G$ there exists some $c \in G$ such that $a * c = \varepsilon$.

The element ε in (G2) is called a *two-sided identity*. The element b in (G3) is called a *two-sided inverse* for a and the element c in (G3) is called a *right inverse* for a.

(a) If (G1) and (G2) hold, prove that the two-sided identity element is unique.
(b) If (G1), (G2) and (G3) hold, prove that the two-sided inverse is unique.
(c) Assuming that (G1) and (G2) hold, prove that that (G3) and (G4) are equivalent. [Hint: One direction is obvious. The hard part is to prove that the existence of right inverses implies the existence of two-sided inverses.]

2. Groups of Matrices. Matrix multiplication is necessarily associative because it corresponds to composition of linear functions. You may recall from linear algebra that a real $n \times n$ matrix $A \in \text{Mat}_n(\mathbb{R})$ has a unique two-sided inverse precisely when $\det A \neq 0$. Now consider the following sets of matrices:

$$\begin{aligned} GL_n(\mathbb{R}) &= \{A \in \text{Mat}_n(\mathbb{R}) : \det A \neq 0\} \\ SL_n(\mathbb{R}) &= \{A \in \text{Mat}_n(\mathbb{R}) : \det A = 1\} \\ O_n(\mathbb{R}) &= \{A \in \text{Mat}_n(\mathbb{R}) : A^T A = I\} \\ SO_n(\mathbb{R}) &= \{A \in \text{Mat}_n(\mathbb{R}) : A^T A = I \text{ and } \det A = 1\}. \end{aligned}$$

Prove carefully that each one of these sets is a group under matrix multiplication. [Hint: It is helpful to remember that $\det(AB) = \det(A)\det(B)$ and $(AB)^T = B^T A^T$ for all matrices $A, B \in \text{Mat}_n(\mathbb{R})$.]

3. Groups of Permutations. Let S_3 be the set of all permutations of the set $\{1,2,3\}$, i.e., all invertible functions

$$f : \{1,2,3\} \to \{1,2,3\}.$$

(a) List all 6 elements of the set. [I recommend using cycle notation.]
(b) We can think of $(S_3, \circ, \text{id})$ as a group, where $\circ$ is functional composition and id is the identity function. Write out the full 6×6 group table.
(c) Let S_n be the group of permutations of $\{1,2,\ldots,n\}$. An element of S_n is called a *transposition* if it switches two elements of the set and sends every other element to itself. We denote the transposition that switches $i \leftrightarrow j$ by $(ij) \in S_n$. Let $A_n \subseteq S_n$ be the subset of permutations that can be expressed as a composition of an **even** number of transpositions. Prove that $A_n \subseteq S_n$ is a subgroup.
(d) List all elements of the subgroup $A_3 \subseteq S_3$ and draw its group table.