Math 561/661 **Fall 2021**

Homework 6 **Drew Armstrong**

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1. Invariance of Quotient and Remainder. For every extension of fields $\mathbb{E} \supseteq \mathbb{F}$ we obtain an extension of rings $\mathbb{E}[x] \supseteq \mathbb{F}[x]$.

(a) Consider $f(x), g(x) \in \mathbb{F}[x]$ with $g(x) \neq 0$. Since $\mathbb{F}[x] \subseteq \mathbb{E}[x]$ we also have $f(x), g(x) \in \mathbb{E}[x]$, hence there exist some $q(x), r(x) \in \mathbb{E}[x]$ satisfying

$$\begin{cases} f(x) = g(x)q(x) + r(x), \\ r(x) = 0 \text{ or } \deg(r) < \deg(g). \end{cases}$$

Prove that we must in fact have $q(x), r(x) \in \mathbb{F}[x]$. [Hint: Uniqueness.]

(b) Consider any $f(x), g(x) \in \mathbb{F}[x]$. We can also view $f(x), g(x)$ as elements of $\mathbb{E}[x]$. If $f(x)|g(x)$ in the ring $\mathbb{E}[x]$, use part (a) to prove that $f(x)|g(x)$ in the ring $\mathbb{F}[x]$.

(c) Now consider the field extension $\mathbb{C} \supseteq \mathbb{R}$. If $f(x) \in \mathbb{R}[x]$ and $f(i) = 0$, prove that there exists $q(x) \in \mathbb{R}[x]$ such that $f(x) = (x^2+1)q(x)$. [Hint: Use Descartes' Theorem to prove that $(x^2+1)|f(x)$ in the ring $\mathbb{C}[x]$. Then use part (b).]

2. Field of Fractions. Let R be an integral domain and consider the set of abstract fractions

$$\mathrm{Frac}(R) = \{a/b : a, b \in R, b \neq 0\}.$$

We declare that $a/b = a'/b'$ if and only if $ab' = a'b$ and we define the following operations:

$$\frac{a}{b} \cdot \frac{c}{d} := \frac{ac}{bd}$$
$$\frac{a}{b} + \frac{c}{d} := \frac{ad+bc}{bd}.$$

Note that the fractions on the right exist because $b \neq 0$ and $d \neq 0$ imply $bd \neq 0$. One can check that these operations make $\mathrm{Frac}(R)$ in a field with identity elements $0/1$ and $1/1$.

(a) If $a/b = a'/b'$ and $c/d = c'/d'$, prove that

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a'}{b'} \cdot \frac{c'}{d'} \quad \text{and} \quad \frac{a}{b} + \frac{c}{d} = \frac{a'}{b'} + \frac{c'}{d'}.$$

(b) Prove that the function $\varphi : R \to \mathrm{Frac}(R)$ defined by $\varphi(a) := a/1$ is an injective ring homomorphism.

(c) Let $\mathbb{F}$ be a field containing R as a subring. Prove that the function $\mu : \mathrm{Frac}(R) \to \mathbb{F}$ defined by $\mu(a/b) := ab^{-1}$ is an injective ring homomorphism. [Hint: In addition to the usual properties of an injective ring homomorphism, you must also show that $a/b = a'/b'$ implies $\mu(a/b) = \mu(a'/b')$. That is, you must show that μ is "well-defined".]

Remark: Given rings R and S, a *ring homomorphism* is a function $\varphi : R \to S$ satisfying

- $\varphi(1) = 1$,
- $\varphi(a+b) = \varphi(a) + \varphi(b)$,
- $\varphi(ab) = \varphi(a)\varphi(b)$.