Math 561/661 **Fall 2021**
Homework 1 **Drew Armstrong**

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1. Working with Ring Axioms. Let $(R, +, \cdot, 0, 1)$ be a ring.[1] Recall that for any element $a \in R$ there exists a unique element $-a \in R$ such that $a + (-a) = 0$.

(a) Show that $0a = 0$. [Hint: Multiply both sides of $0 + 0 = 0$ by a.]
(b) Show that $-(-a) = a$. [Hint: Uniqueness.]
(c) Show that $a(-b) = (-a)b = -(ab)$. [Hint: Multiply both sides of $b + (-b) = 0$ by a.]
(d) Show that $(-a)(-b) = ab$. [Hint: Combine parts (b) and (c).]

2. Complex Conjugation. Given a complex number $\alpha = a + bi \in \mathbb{C}$ we define the complex conjugate by $\alpha^* = a - bi$.

(a) For all $\alpha \in \mathbb{C}$ show that $\alpha^* = \alpha$ if and only if $\alpha \in \mathbb{R}$.
(b) For all $\alpha, \beta \in \mathbb{C}$ show that $(\alpha + \beta)^* = \alpha^* + \beta^*$ and $(\alpha\beta)^* = \alpha^*\beta^*$.
(c) If $f(x) \in \mathbb{R}[x]$ is a polynomial with real coefficients, show that the non-real complex roots of f come in conjugate pairs. [Hint: For all $\alpha \in \mathbb{C}$ show that $f(\alpha)^* = f(\alpha^*)$.]

3. Absolute Value of Complex Numbers. Given a complex number $\alpha = a + bi \in \mathbb{C}$ we define the absolute value by $|\alpha| = +\sqrt{a^2 + b^2}$.

(a) Show that $\alpha = 0$ if and only if $|\alpha| = 0$. [Hint: For all $a \in \mathbb{R}$ we have $a^2 \geq 0$.]
(b) Show that $\alpha\alpha^* = |\alpha|^2$.
(c) For all $\alpha, \beta \in \mathbb{C}$ show that $|\alpha\beta| = |\alpha||\beta|$. [Hint: Part (b) gives a shortcut.]
(d) For all $\alpha, \beta \in \mathbb{C}$ show that $\alpha\beta = 0$ implies $\alpha = 0$ or $\beta = 0$. [Hint: Use parts (a,c).]

4. Descartes' Factor Theorem. Let $\mathbb{F}$ be a field and let $\mathbb{F}[x]$ be the ring of polynomials

$$\mathbb{F}[x] = \{a_0 + a_1 x + \cdots + a_n x^n : a_0, \ldots, a_n \in \mathbb{F}, n \geq 0\}.$$

If $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ with $a_n \neq 0$ then we write $\deg(f) = n$. The zero polynomial does not have a degree.

(a) Show that $\deg(fg) = \deg(f) + \deg(g)$ for all nonzero polynomials $f(x), g(x) \in \mathbb{F}[x]$.
(b) Suppose that a nonzero polynomial $f(x) \in \mathbb{F}[x]$ satisfies $f(\alpha) = 0$ for some $\alpha \in \mathbb{F}$. In this case prove that we have $f(x) = (x - \alpha)g(x)$ for some polynomial $g(x)$ with $\deg(g) = \deg(f) - 1$. [Hint: By long division there exist polynomials $q(x), r(x) \in \mathbb{F}[x]$ with $f(x) = (x - \alpha)q(x) + r(x)$, such that $r(x)$ is a constant.]
(c) Use part (b) to prove that a polynomial $f(x) \in \mathbb{F}[x]$ of degree n has **at most** n **distinct roots in** $\mathbb{F}$. [Hint: If $f(\alpha) = 0$ then $f(x) = (x - \alpha)g(x)$ for some polynomial of degree $n - 1$. What happens if $f(\beta) = 0$ for some $\beta \neq \alpha$? Use induction.]

5. Leibniz' Mistake. In 1702 Gottfried Leibniz claimed that the polynomial $x^4 + 1$ cannot be factored as a product of smaller polynomials with real coefficients.

(a) Use the polar form to find all of the complex 4th roots of -1.
(b) Use this to factor the polynomial $x^4 + 1$ and show that Leibniz was wrong. [Hint: Group the four roots into complex conjugate pairs.]

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[1] We always assume that a ring has commutative multiplication.