Math 561/661 **Fall 2018**
Homework 2 **Drew Armstrong**

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1. Powers of a Cycle. Consider the standard 12-cycle in cycle notation:

$$c := (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12) \in S_{12}.$$

Compute the first twelve powers $c, c^2, c^3 \ldots, c^{12}$ and express each of them in cycle notation. Try to guess what the k-th power of an n-cycle looks like.

2. Homomorphism and Isomorphism. Let $(G, *, \delta)$ and $(H, \bullet, \varepsilon)$ be abstract groups and let $f : G \to H$ be a function. We say that f is a *(group) homomorphism* if it satisfies the following condition:

$$\text{for all } a, b \in G \text{ we have } f(a * b) = f(a) \bullet f(b).$$

(a) If $f : G \to H$ is a homomorphism, prove that $f(\delta) = \varepsilon$.
(b) If $f : G \to H$ is a homomorphism, prove that $f(a^{-1}) = f(a)^{-1}$ for all $a \in G$.
(c) Suppose that $f : G \to H$ is homomorphism and that the inverse function exists. Prove that the function $f^{-1} : H \to G$ is also a homomorphism. It follows that invertible homomorphisms are the same as isomorphisms.

3. Isometries = Orthogonal Matrices. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ be column vectors and let $\mathbf{x}^T$ denote the row vector corresponding to $\mathbf{x}$. We define the standard inner product as follows:

$$\langle \mathbf{x}, \mathbf{y} \rangle := \mathbf{x}^T \mathbf{y} = \sum_i x_i y_i.$$

Recall the the distance between two points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ is defined by $\|\mathbf{x} - \mathbf{y}\|^2 = \langle \mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle$ and recall that the following properties are satisfied:

- We have $\|\mathbf{x}\|^2 = \langle \mathbf{x}, \mathbf{x} \rangle = 0$ if and only if $\mathbf{x} = \mathbf{0}$.
- For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ we have $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$.
- For all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$ we have $\langle \mathbf{x}, \alpha\mathbf{y} + \beta\mathbf{z} \rangle = \alpha\langle \mathbf{x}, \mathbf{y} \rangle + \beta\langle \mathbf{x}, \mathbf{z} \rangle$.

The goal of this problem is to show the following: If $f : \mathbb{R}^n \to \mathbb{R}^n$ is any function that preserves distance and sends the origin to itself then it preserves the inner product. Hence the function is linear. Hence we have $f(\mathbf{x}) = A\mathbf{x}$ for some $n \times n$ matrix A, which must satisfy $A^T A = I$.

(a) Assume that the function $f : \mathbb{R}^n \to \mathbb{R}^n$ preserves the distance between any two points (i.e., $\|f(\mathbf{x}) - f(\mathbf{y})\|^2 = \|\mathbf{x} - \mathbf{y}\|^2$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$) and sends the origin to iself (i.e., $f(\mathbf{0}) = \mathbf{0}$). Prove that

$$\langle f(\mathbf{x}), f(\mathbf{y}) \rangle = \langle \mathbf{x}, \mathbf{y} \rangle \quad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

(b) Continuing from part (a), prove that this f is a linear function. [Hint: For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ show that

$$\|f(\mathbf{x} + \mathbf{y}) - (f(\mathbf{x}) + f(\mathbf{y}))\|^2 = 0 \qquad \text{and} \qquad \|f(\alpha\mathbf{x}) - \alpha f(\mathbf{x})\|^2.]$$

(c) Continuing from (a) and (b), show that $f(\mathbf{x}) = A\mathbf{x}$ for some $n \times n$ matrix satisfying $A^T A = I$. [Hint: Let $\mathbf{e}_1, \ldots, \mathbf{e}_n \in \mathbb{R}^n$ be the standard basis vectors. Then $f(\mathbf{e}_i)$ is the i-th column of A. To show that $A^T A = I$ use the fact that $\mathbf{e}_i^T B \mathbf{e}_j$ is equal to the i, j-entry of an arbitrary matrix B.]

4. Rotation and Reflection. In class I showed that every element of $O_2(\mathbb{R})$ has the form

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \text{or} \qquad F_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}.$$

(a) Verify that $R_\theta \in SO_2(\mathbb{R})$ and that $F_\theta \in O_2(\mathbb{R}) - SO_2(\mathbb{R})$.
(b) We saw on the previous homework that $\mathbf{x} \mapsto R_\theta \mathbf{x}$ is a rotation. Use a similar argument to prove that $\mathbf{x} \mapsto F_\theta \mathbf{x}$ is a reflection.
(c) For all $\alpha, \beta \in \mathbb{R}$ prove that
- $R_\alpha R_\beta = R_{\alpha+\beta}$,
- $F_\alpha F_\beta = R_{\alpha-\beta}$,
- $R_\alpha F_\beta = F_\beta (R_\alpha)^{-1} = F_{\alpha+\beta}$.

(d) Fix a positive integer n and define the matrices $R := R_{2\pi/n}$ and $F := F_0$. The subgroup of $O_2(\mathbb{R})$ generated by the set $\{R, F\}$ has $2n$ elements. Use (c) to find them all.

5. The Fermat-Euler-Lagrange Theorem. Let $(G, *, \varepsilon)$ be a group and let $g \in G$ be any element. Define the function $f_g : G \to G$ by $f_g(a) := g * a$.

(a) Prove that $f_g : G \to G$ is a bijection.
(b) If G is a **finite abelian** group, prove that $g^{\#G} = \varepsilon$. [Hint: Suppose that $G = \{a_1, a_2, \ldots, a_n\}$. Explain why $\prod_i a_i = \prod_i f_g(a_i)$. Rearrange and then cancel.]

[Remark: This theorem is also true for **finite non-abelian** groups but we don't have the technology to prove it yet.]

6. Join of Two Subgroups. Let G be a group and let $H, K \subseteq G$ be subgroups. Recall that the subgroup generated by the union $H \cup K$ is called the *join*:

$$H \vee K := \langle H \cup K \rangle = \text{ the intersection of all subgroups that contain } H \cup K.$$

(a) If $(G, +, 0)$ is abelian, we define the *sum* of H and K as follows:

$$H + K := \{h + k : h \in H, k \in K\}.$$

Prove that this is a subgroup.

(b) If $(G, +, 0)$ is abelian, use part (a) to prove that $H \vee K = H + K$.
(c) If $(G, *, \varepsilon)$ is non-abelian, show that the following set is **not** necessarily a subgroup, and hence it does not coincide with the join:

$$H * K := \{h * k : h \in H, k \in K\}.$$

[Hint: The smallest non-abelian group is S_3.]