MTH 561/661 **Fall 2018**
Exam 3 — Mon Dec 3 **Drew Armstrong**

Problem 1. Multiplication of Subgroups. Let $H, K \subseteq G$ be subgroups and consider the multiplication function $\mu : H \times K \to G$ defined by $\mu(h, k) := hk$.

(a) Prove that μ is injective if and only if $H \cap K = \{\varepsilon\}$.

Proof. First suppose that μ is injective and let $g \in H \cap K$. Then since

$$\mu(g, g^{-1}) = \varepsilon = \mu(\varepsilon, \varepsilon)$$

we have $(g, g^{-1}) = (\varepsilon, \varepsilon)$, and hence $g = \varepsilon$. Conversely, let $H \cap K = \{\varepsilon\}$ and suppose that $\mu(h_1, k_2) = \mu(h_2, k_2)$. Then we have

$$h_1 k_1 = h_2 k_2$$
$$h_2^{-1} h_1 = k_2 k_1^{-1} \in H \cap K,$$

which implies that $h_2^{-1} h_1 = \varepsilon$ and $k_2 k_1^{-1} = \varepsilon$, hence $h_1 = h_2$ and $k_1 = k_2$. □

(b) If G is abelian, prove that $\operatorname{im} \mu \subseteq G$ is a subgroup.

Proof. Suppose that G is abelian. Then for all elements $h_1 k_1$ and $h_2 k_2$ in $\operatorname{im} \mu$ we have

$$(h_1 k_1)(h_2 k_2)^{-1} = (h_1 k_1)(k_2^{-1} h_2^{-1}) = (h_1 h_2^{-1})(k_1 k_2^{-1}) \in \operatorname{im} \mu.$$

□

Problem 2. Direct Product. Let $H, K \subseteq G$ be subgroups.

(a) Suppose that $\gcd(\#H, \#K) = 1$ and use this to prove that $H \cap K = \{\varepsilon\}$.

Proof. Since $H \cap K \subseteq H$ is a subgroup, Lagrange's Theorem says that $\#(H \cap K) | \#H$. Similarly, we have $\#(H \cap K) | \#K$. Then since $\gcd(\#H, \#K) = 1$ we conclude that $\#(H \cap K) = 1$, and hence $H \cap K = \{\varepsilon\}$. □

(b) Assume also that G is abelian with $\#G = \#H \cdot \#K$. Use this to prove that $G = H \times K$.

Proof. From (a) and Problem 1 we know that $\mu : H \times K \to G$ is injective, hence

$$\#(\operatorname{im} \mu) = \#(H \times K) = \#H \cdot \#K = \#G.$$

Since $\operatorname{im} \mu \subseteq G$, this implies that $G = \operatorname{im} \mu$. Finally, since G is abelian we know that $H \trianglelefteq G$ and $K \trianglelefteq G$. □

Problem 3. Orbit-Stabilizer Theorem. Let X be a "set with structure" and let $\varphi : G \to \operatorname{Aut}(X)$ be a group homomorphism. For all $x \in X$ we define

$$\operatorname{Orb}_\varphi(x) := \{\varphi_g(x) : g \in G\} \subseteq X,$$
$$\operatorname{Stab}_\varphi(x) := \{g \in G : \varphi_g(x) = x\} \subseteq G.$$

You can assume that $\varphi_\varepsilon = \operatorname{id}$ and $\varphi_g^{-1} = \varphi_{g^{-1}}$ for all $g \in G$.

(a) For all $x \in X$, prove that $\operatorname{Stab}_\varphi(x) \subseteq G$ is a subgroup.

Proof. Fix an element $x \in X$.

- **Identity.** Since $\varphi_\varepsilon(x) = \mathrm{id}(x) = x$ we have $\varepsilon \in \mathrm{Stab}_\varphi(x)$.
- **Inverse.** For any $g \in \mathrm{Stab}_\varphi(x)$ we have

$$\varphi_g(x) = x \quad \Longrightarrow \quad x = \varphi_{g^{-1}}(x),$$

  and hence $g^{-1} \in \mathrm{Stab}_\varphi(x)$.
- **Closure.** For any $g, h \in \mathrm{Stab}_\varphi(x)$ we have

$$\varphi_{gh}(x) = (\varphi_g \circ \varphi_h)(x) = \varphi_g(\varphi_h(x)) = \varphi_g(x) = x,$$

  and hence $gh \in \mathrm{Stab}_\varphi(x)$.

□

(b) For all $x \in X$, prove that the rule $\varphi_g(x) \mapsto g \cdot \mathrm{Stab}_\varphi(x)$ defines a bijection from points of the orbit to left cosets of the stabilizer:

$$\mathrm{Orb}_\varphi(x) \to G/\mathrm{Stab}_\varphi(x).$$

Proof. The map is clearly surjective. It is well-defined and injective since for all $g, h \in G$ we have

$$\begin{aligned}
\varphi_g(x) = \varphi_h(x) &\Longleftrightarrow \varphi_{g^{-1}}(\varphi_h(x)) = x \\
&\Longleftrightarrow \varphi_{g^{-1}h}(x) = x \\
&\Longleftrightarrow g^{-1}h \in \mathrm{Stab}_\varphi(x) \\
&\Longleftrightarrow g \cdot \mathrm{Stab}_\varphi(x) = h \cdot \mathrm{Stab}_\varphi(x).
\end{aligned}$$

□

Problem 4. Semidirect Product. Let Isom be the group of isometries $f : \mathbb{R}^n \to \mathbb{R}^n$. For all $\mathbf{u} \in \mathbb{R}^n$ you can assume that the translation $\tau_\mathbf{u}(\mathbf{x}) := \mathbf{x} + \mathbf{u}$ is an isometry, and for all $f \in \mathrm{Isom}$ you can assume that $f(\mathbf{0}) = \mathbf{0}$ implies $f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Consider the subgroups

$$\begin{aligned}
T &:= \{\tau_\mathbf{u} : \mathbf{u} \in \mathbb{R}^n\} \subseteq \mathrm{Isom}, \\
\mathrm{Isom}_\mathbf{0} &:= \{f \in \mathrm{Isom} : f(\mathbf{0}) = \mathbf{0}\} \subseteq \mathrm{Isom}.
\end{aligned}$$

(a) Prove that every $f \in \mathrm{Isom}$ has the form $f = \tau_\mathbf{u} \circ g$ for some $\tau_\mathbf{u} \in T$ and $g \in \mathrm{Isom}_\mathbf{0}$.

Proof. Suppose that $f(\mathbf{0}) = \mathbf{u}$ and define the isometry $g := \tau_{-\mathbf{u}} \circ f$, so that $f = \tau_\mathbf{u} \circ g$. Then f has the correct form because

$$g(\mathbf{0}) = (\tau_{-\mathbf{u}} \circ f)(\mathbf{0}) = \tau_{-\mathbf{u}}(f(\mathbf{0})) = \tau_{-\mathbf{u}}(\mathbf{u}) = \mathbf{u} - \mathbf{u} = \mathbf{0},$$

and hence $g \in \mathrm{Isom}_\mathbf{0}$. □

(b) For all $\tau_\mathbf{u} \in T$ and $f \in \mathrm{Isom}_\mathbf{0}$, prove that $f \circ \tau_\mathbf{u} \circ f^{-1} \in T$.

Proof. For all $\mathbf{x} \in \mathbb{R}^n$ we have

$$\begin{aligned}
(f \circ \tau_\mathbf{u} \circ f^{-1})(\mathbf{x}) &= f(\tau_\mathbf{u}(f^{-1}(\mathbf{x}))) \\
&= f(f^{-1}(\mathbf{x}) + \mathbf{u}) \\
&= f(f^{-1}(\mathbf{x})) + f(\mathbf{u}) \\
&= \mathbf{x} + f(\mathbf{u}).
\end{aligned}$$

It follows that $f \circ \tau_\mathbf{u} \circ f^{-1} = \tau_{f(\mathbf{u})} \in T$ as desired. □