Math 561 H **Fall 2011**
Homework 2 **Drew Armstrong**

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1. Let $\varphi : G \to H$ be a homomorphism of groups. Prove that $\operatorname{im}\varphi$ is a subgroup of H.

2. Let G be a set with binary operation $(a, b) \mapsto ab$ and consider the following possible axioms:

(1) $\forall\, a, b \in G, a(bc) = (ab)c$.
(2) $\exists\, e \in G, \forall\, a \in G, ae = ea = a$.
(3) $\forall\, a \in G, \exists\, b \in G, ab = ba = e$.
(3') $\forall\, a \in G, \exists\, b \in G, ab = e$.

Prove that the axioms (3) and (3') are equivalent. That is, show that (1), (2), and (3) hold if and only if (1), (2), and (3') hold. (One direction is easy. For the other direction, let $a \in G$. Then there exist $b, c \in G$ such that $ab = e$ and $bc = e$. Show that $a = c$.)

3. Let H, K be subgroups of G. Prove that $H \cap K$ is also a subgroup of G.

4. (a) Consider a homomorphism $\varphi : \mathbb{Z}^+ \to G$ with $\varphi(1) = g \in G$. Describe $\operatorname{im}\varphi$ and $\ker\varphi$.
(b) Describe the set of **automorphisms** $\varphi : \mathbb{Z}^+ \to \mathbb{Z}^+$.

5. Given a group G, define its center:

$$Z(G) := \{g \in G : \forall\, h \in G, gh = hg\}.$$

Prove that $Z(G)$ is a normal subgroup of G. (We write $Z(G) \trianglelefteq G$.)

6. Prove that the center of $GL_n(\mathbb{R})$ is the set $\{cI : c \in \mathbb{R}, c \neq 0\}$ of "scalar matrices". (Hint: Let $E_{i,j}$ be the matrix with 1 in its i, j-position and zeroes elsewhere. What does $AE_{i,j} = E_{i,j}A$ mean? What does $A(E_{i,j} + E_{j,i}) = (E_{i,j} + E_{j,i})A$ mean?)

7. Consider the matrix $R_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

(a) Given $\mathbf{x} \in \mathbb{R}^2$ show that $R_\theta\,\mathbf{x}$ is the rotation of $\mathbf{x}$ by θ degrees counterclockwise. (Hint: It suffices to let $\mathbf{x} = \mathbf{e}_1$ and $\mathbf{x} = \mathbf{e}_2$.)
(b) If $A \in SO_2(\mathbb{R})$ prove that $A = R_\theta$ for some $\theta \in \mathbb{R}$.
(c) Verify that the map $\varphi(e^{i\theta}) := R_\theta$ is an isomorphism $U(1) \approx SO_2(\mathbb{R})$.

8. Given $a, b \in G$ we say that a and b are conjugate if there exists $g \in G$ such that $a = gbg^{-1}$. **Prove** that conjugacy is an equivalence relation on G. (The equivalence classes are called conjugacy classes.) **Prove**: If $a, b \in G$ are conjugate then they have the same order.