Example: Find all left & right inverses for matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}.$$

Right inverse

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A\left(\vec{b}_1 \ \vec{b}_2\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\left(A\vec{b}_1 \,\middle|\, A\vec{b}_2\right) = \left(\begin{matrix} 1 \\ 0 \end{matrix} \,\middle|\, \begin{matrix} 0 \\ 1 \end{matrix}\right)$$

$$\begin{cases} A\vec{b}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ A\vec{b}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{cases}$$

$$A\vec{b}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightsquigarrow \vec{b}_1 = \begin{pmatrix} 2+s \\ -1-2s \\ s \end{pmatrix}$$

$$A\vec{b}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightsquigarrow \vec{b}_2 = \begin{pmatrix} -1+t \\ 1-2t \\ t \end{pmatrix}.$$

Two-dimensional family of right inv:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 2+s & -1+t \\ -1-2s & 1-2t \\ s & t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

For <u>any</u> s, t.

Not a subspace of $\mathbb{R}^{3\times 2}$

∘ doesn't contain $\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Left inverses?

$$CA = I$$

$$\begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This is not possible.

<u>Proof</u>: We have at least two right inverses, $B \neq B'$.

<u>If</u> A had a left inverse C

then

$$
\begin{aligned}
I_2 &= I_2 \\
AB &= AB' \\
C(AB) &= C(AB') \\
(CA)B &= (CA)B' \\
I_3 B &= I_3 B' \\
B &= B' \quad \times .
\end{aligned}
$$

Goal: Study two-sided inverses:

$$AB = I \quad \& \quad BA = I.$$

when does B exist?

How to find it?

Need to discuss subspaces associated to linear transformations & matrices.

Consider a linear trans

$$T : V \to W$$

Call this an isomorphism of vector spaces when it is

- injective (one-to-one)
- surjective (onto)

If we can find an isomorphism $V \to W$ then we write

$$V \cong W$$

isomorphic as vector spaces.

Key Fact:

$$V \cong W \implies \dim V = \dim W$$

Proof: Let $T : V \to W$ be an isomorphism.

Let $\vec{b}_1, \dots, \vec{b}_n \in V$ be a basis.

Claim : $T(\vec{b}_1), \dots, T(\vec{b}_n) \in W$

is a basis of W.

What do we need ?

- Spanning set.

Given $\vec{w} \in W$ need

$$\vec{w} = ?\,T(\vec{b}_1) + \dots + ?\,T(\vec{b}_n).$$

Well, T is <u>onto</u> we can find some $\vec{v} \in V$ such that $\vec{w} = T(\vec{v})$.

Since $\vec{b}_1, \dots, \vec{b}_n$ spans V have

$$\vec{v} = a_1 \vec{b}_1 + \dots + a_n \vec{b}_n$$

$$\vec{w} = T(\vec{v}) = a_1 T(\vec{b}_1) + \dots + a_n T(\vec{b}_n)$$

✓ ✓

- Independent ?

$$a_1 T(\vec{b}_1) + \dots + a_n T(\vec{b}_n) = \vec{0}.$$

Need to show $a_1 = a_2 = \dots = a_n = 0$.

[T one-to-one equivalent to

$$T(\vec{v}) = \vec{0} \iff \vec{v} = \vec{0}.$$

Proof: T one-to-one then

$$T(\vec{v}) = \vec{0}$$

$$\Longrightarrow T(\vec{v}) = T(\vec{0})$$ one-to-one.

$$\Longrightarrow \vec{v} = \vec{0}.$$

Conversely, if

$$T(\vec{v}) = \vec{0} \Longrightarrow \vec{v} = 0.$$

Then

$$T(\vec{v}_1) = T(\vec{v}_2)$$

$$T(\vec{v}_1) - T(\vec{v}_2) = \vec{0}$$

$$T(\vec{v}_1 - \vec{v}_2) = \vec{0}$$

$$\vec{v}_1 - \vec{v}_2 = \vec{0}$$

$$\vec{v}_1 = \vec{v}_2 \quad \checkmark \quad]$$

$$\vec{0} = a_1 T(\vec{b}_1) + \cdots + a_n T(\vec{b}_n)$$

$$\vec{0} = T(a_1\vec{b}_1 + \cdots + a_n\vec{b}_n)$$ one-to-one.

$$\vec{0} = a_1\vec{b}_1 + \cdots + a_n\vec{b}_n$$

$\Longrightarrow a_1 = a_2 = \cdots = a_n = 0$
because $\vec{b}_1, \ldots, \vec{b}_n$ are independent. ✓

Conclusion:

$$V \cong W \Longrightarrow \dim V = \dim W.$$

The Fundamental Theorem:

Given matrix A,

$$\dim R(A) = \dim C(A).$$

Strategy:

(1) Show for any invertible matrices E & F that

$$\dim R(A) = \dim R(EAF).$$
$$\dim C(A) = \dim C(EAF).$$

② Find E & F to put A in a simple form:

$$EAF = \begin{array}{c} \\ r\{ \end{array} \left(\begin{array}{c|c} \overbrace{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}}^{r} & 0 \\ \hline 0 & 0 \end{array} \right)$$

Then $\dim C(EAF) = r$ ✓ :)
$\dim R(EAF) = r$

Proof of ①: For inv. E & F

We will show:

$$R(EA) = R(A)$$
$$R(AF) \cong R(A)$$

not equal but isom.

Idea: Let $\vec{a}_1^T \cdots \vec{a}_m^T$ be the rows of A.

$$A = \begin{pmatrix} — & \vec{a}_1^T & — \\ \vdots & & \vdots \\ — & \vec{a}_m^T & — \end{pmatrix}$$

Write $E = \begin{pmatrix} e_{11} & \cdots & e_{1m} \\ \vdots & & \vdots \\ e_{m1} & \cdots & e_{mm} \end{pmatrix}$

$$i\text{th row } EA = (i\text{th row } E)\, A$$

$$= \begin{pmatrix} e_{i1} & \cdots & e_{im} \end{pmatrix} \begin{pmatrix} - & \vec{a}_1^{\,T} & - \\ & \vdots & \\ - & \vec{a}_m^{\,T} & - \end{pmatrix}$$

$$= \underbrace{e_{i1}\vec{a}_1^{\,T} + \cdots + e_{im}\vec{a}_m^{\,T}}_{\text{lin. comb. of rows of } A.}.$$

Each row of EA is in $R(A)$.

Hence $R(EA) \leq R(A)$.

[Technically:

$$R(B) = \operatorname{span}\{\text{rows of } B\}.$$

We showed every lin comb of rows of EA is also a lin comb of rows of A:

$$\left. R(EA) \subseteq R(A). \right]$$

Now (trick) since E^{-1} is invertible ($(E^{-1})^{-1} = E$) have

$$R(E^{-1}B) \subseteq R(B) \quad \text{any } B.$$

$$R(E^{-1}EA) \subseteq R(EA)$$

$$R(A) \subseteq R(EA).$$

Hence $R(A) = R(EA)$ ✓

Next: $R(AF) \overset{?}{=} R(A)$

Rows of AF?

$$i\text{th row } AF = (i\text{th row } A)F$$

$$\vec{a}_i^T F$$

$$R(A) = \operatorname{span}\{\vec{a}_1^T, \dots, \vec{a}_m^T\}$$

$$R(AF) = \operatorname{span}\{\vec{a}_1^T F, \dots, \vec{a}_m^T F\}$$

Define an isomorphism

$$T : R(A) \longrightarrow R(AF)$$
$$\vec{v}^T \longmapsto \vec{v}^T F$$

$$T\left(a_1 \vec{a}_1^T + \cdots a_n \vec{a}_n^T\right)$$
$$= a_1 T(\vec{a}_1^T) + \cdots + a_m T(\vec{a}_m^T).$$
$$= a_1 (\vec{a}_1^T F) + \cdots + a_m (\vec{a}_m^T F).$$
$$\in R(AF)$$

Linear: $\left(a\vec{v}^T + b\vec{w}^T\right) F$
$$= a\vec{v}^T F + b\vec{w}^T F.$$

Invertible with inverse
$$\vec{w}^T \longmapsto \vec{w}^T F^{-1}$$

[Recall: $T : V \to W$ called isom

- linear
- injective
- surjective

$\left.\begin{matrix}\text{injective}\\ \text{surjective}\end{matrix}\right\}$ bijective]

Conclusion:

$$\dim R(A) = \dim R(AF).$$

Similar proof shows

$$C(EA) \cong C(A)$$

$$C(AF) = C(A).$$

(2) Learn how to perform row & column operations via matrix multiplication.