Preview of HW 2 :

Frobenius norm

$$\|A\|_F = \sqrt{\sum_{i,j} |a_{ij}|^2}$$

$$\|AB\|_F \leq \|A\|_F \|B\|_F$$

Geometric Series: If $\|A\|_F < 1$ then $I - A$ (A square) is invertible with

$$(I-A)^{-1} = I + A + A^2 + \cdots$$

Matrix Exponential: For square A,

$$\exp(A) = I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

converges with respect to $\|\cdot\|_F$.

e.g.

$$\exp\begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

e.g. $$P = \left(\begin{array}{c|c} I & R \\ \hline 0 & Q \end{array}\right)$$

Q square with $\|Q\|_F < 1$.

Theorem:

$$P^n \to \left(\begin{array}{c|c} I & R(I-Q)^{-1} \\ \hline 0 & 0 \end{array}\right)$$

as $n \to \infty$.

We've seen matrix multiplication.
Can we also <u>divide</u>?

Let A be $m \times n$
B be $n \times m$

If $AB = I_m$ we say

- A is a left inverse of B
- B is a right inverse of A

Useful for solving matrix equations:

$$BX = C$$
$$A(BX) = AC$$
$$(AB)X = AC$$
$$IX = AC$$
$$X = AC$$

✓

$$XA = D$$
$$(XA)B = DB$$
$$X(AB) = DB$$
$$XI = DB$$
$$X = DB$$

✓

Say A & B are two-sided inverses if

$$AB = I_m \quad \& \quad BA = I_n$$

Observe: Two sided inverses are <u>unique</u>.

Say
$$AB = I \quad \& \quad AC = I$$
$$BA = I \qquad\;\; CA = I.$$

Then

$$B = BI = B(AC)$$
$$= (BA)C = IC = C \checkmark$$

In this case we write

$$A^{-1} = B \quad \& \quad B^{-1} = A.$$

So far this is just symbolic; we don't know if and when such inverses exist.

More symbolic manipulation:

- If A^{-1} exists then $(A^*)^{-1}$ exists & $(A^*)^{-1} = (A^{-1})^*$.

Proof:

$$AA^{-1} = I$$

$$(AA^{-1})^* = I^*$$

$$(A^{-1})^* A^* = I \quad \checkmark$$

Similarly,

$$A^{-1}A = I \implies A^*(A^{-1})^* = I \quad \checkmark$$

- If A^{-1}, B^{-1} & AB exist then $(AB)^{-1}$ exists and $(AB)^{-1} = B^{-1}A^{-1}$.

A $m \times n$
B $n \times m$
A^{-1} $m \times n$
B^{-1} $n \times m$

Proof: $(AB)(B^{-1}A^{-1})$

$= A(BB^{-1})A^{-1}$

$= AIA^{-1}$

$= AA^{-1}$

$= I$.

Similarly, $(B^{-1}A^{-1})(AB) = I$.

The Big Question:

When does A^{-1} exist, and how to compute it?

Recall 2×2:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

So, A^{-1} exists $\Longleftrightarrow$ $\det A \neq 0$.

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Same ideas hold for all sizes, but formulas are too big to write down.

TWO SUBTLE THEOREMS:

- IF A^{-1} exists, then A is SQUARE!

- Given square A & B,
  $AB = I \Longleftrightarrow BA = I$.

If you think these facts are

easy, go ahead and try to prove them!

e.g. $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$

Assume $AB = I$ so

$$\begin{pmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\begin{cases} aa' + bc' = 1, \\ ab' + bd' = 0, \\ ca' + dc' = 0, \\ cb' + dd' = 1. \end{cases}$$

Use these 4 equations to prove that $BA = I$, i.e., that

$$\begin{cases} a'a + b'c = 1, \\ a'b + b'd = 0, \\ c'a + d'c = 0, \\ c'b + d'd = 1. \end{cases}$$

You won't be able to because this is the wrong approach to the problem.

The actual way to solve this involves the dimensions of certain subspaces of Euclidean space.

I'll follow Gilbert Strang's notation. Given $A \in \mathbb{R}^{m \times n}$, we define 4 subspaces:

$$C(A) = \text{column space} \subseteq \mathbb{R}^m$$

$$R(A) = \text{row space} \subseteq \mathbb{R}^n$$

$$= C(A^T).$$

$$N(A) = \{\vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0}\} \subseteq \mathbb{R}^n$$

$$N(A^T) = \{ \vec{x} \in \mathbb{R}^m : A^T \vec{x} = \vec{0} \} \subseteq \mathbb{R}^m.$$

e.g. $$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}.$$

$$C(A) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\} = \mathbb{R}^2$$

$$R(A) = \underbrace{\operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}}_{\text{2D plane}} \subseteq \mathbb{R}^3$$

$$N(A) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

= solutions of linear system

$$\begin{cases} x + y + z = 0, \\ x + 2y + 3z = 0. \end{cases}$$

= intersection of 2 non-parallel planes in $\mathbb{R}^3$, which is a line.

$$N(A^T) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

= Solutions of linear system

$$\begin{cases} x + y = 0, \\ x + 2y = 0, \\ x + 3y = 0. \end{cases}$$

= The trivial subspace

$$\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \subseteq \mathbb{R}^2,$$

which has dimension zero.

Fundamental Theorem (Strang).

$$\underbrace{\dim C(A) = \dim R(A)}.$$

common dimension is called the <u>rank</u> of A.

Proof is algorithmic using reduced row echelon form.