HW 4 due Mon.

Exam 2 on FRI Nov 4.

Recall: A "determinant function" φ sends $n\times n$ matrices to $\mathbb{R}$ and satisfies 3 rules:

(1) n-Multilinear in the columns

(2) Alternating

(3) Normalized: $\varphi(I_n) = 1$.

Lemmas:

(A) A has repeated column

$$\Longrightarrow \varphi(A) = 0.$$

(B) A has <u>dependent</u> cols.

$$\Longrightarrow \varphi(A) = 0.$$

HW 4

Hint: W.L.O.G. $\vec{a}_1 = b_2\vec{a}_2 + \cdots + b_n\vec{a}_n$

Let $\hat{A}_i(\vec{v})$ be matrix obtained from a by replacing its ith column with $\vec{v}$:

$$\hat{A}_i(\vec{v}) = \left(\vec{a}_1 \,|\, \cdots \,|\, \vec{a}_{i-1} \,|\, \vec{v} \,|\, \vec{a}_{i+1} \,|\, \cdots \,|\, \vec{a}_n \right)$$

By multilinearity:

$$\varphi(A) = b_2\, \varphi(\cancel{\hat{A}_1(\vec{a}_2)})^{0} + \cdots + b_n\, \varphi(\cancel{\hat{A}_1(\vec{a}_n)})^{0}$$

each of these has a repeated col.

(C) Elementary Matrices:

$$\varphi(A\, D_i(\lambda)) = \lambda\, \varphi(A)$$

$$\varphi(A\, L_{ij}(\lambda)) = \varphi(A).$$

$$\varphi(A\, T_{ij}) = -\varphi(A).$$

(D) Put $A = I$ in (C):

$$\varphi(D_i(\lambda)) = \lambda$$

$$\varphi(L_{ij}(\lambda)) = 1$$

$\varphi(T_{ij}) = -1$.

(E) For elementary $E_1, \dots, E_k$,

$$\varphi(E_1 \cdots E_k) = \varphi(E_1) \cdots \varphi(E_k).$$

Proof: Combine (C) & (D):

$$\varphi(AE) = \varphi(A) \cdot \varphi(E).$$

Induction:

$$\varphi(\underbrace{E_1 \cdots}_{A} E_k) = \varphi(\underbrace{E_1 \cdots E_{k-1}}_{A})\varphi(E_k)$$

$$\vdots$$

$$= \varphi(E_1) \cdots \varphi(E_k).$$

Theorem: Let δ_1 & δ_2 be determinant functions. Then

$$\delta_1 = \delta_2$$

Proof: A not invertible

$\Longrightarrow$ A dependent cols

$\Longrightarrow$ $\delta_1(A) = 0 = \delta_2(A)$.

A <u>invertible</u>. Factor

$$A = \underbrace{E_1 \cdots E_k}_{\text{row or col reduction}}.$$

Then
$$\begin{aligned} \delta_1(A) &= \delta_1(E_1) \cdots \delta_1(E_k) \\ &= \delta_2(E_1) \cdots \delta_2(E_k) \\ &= \delta_2(A) \quad \checkmark \end{aligned}$$

Idea: Define det by

- Factoring $A = E_1 \cdots E_k$
- Set $\det(A) = \underbrace{\det(E_1) \cdots \det(E_k)}_{\text{easy.}}$

Problem: Factorization is not unique.

Different Factorizations $\overset{?}{\Longrightarrow}$ Different values of det

The standard definition of det

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \cdot a_{\sigma(1),1} \, a_{\sigma(2),2} \cdots a_{\sigma(n),n}$$

What ?

S_n set of permutations

$$\sigma : \{1, 2, \dots, n\} \longrightarrow \{1, \dots, n\}$$

bijective.

$$\sigma = (\sigma(1), \sigma(2), \dots, \sigma(n)).$$

$$S_3 = \{ (1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1) \}$$

$\# S_n = n!$

The "sign" or "parity" of a permutation.

How to get from $(1, 2, \dots, n) \to \sigma$?

e.g. $\sigma = (3,2,1)$

$$(1,2,3) \to (2,1,3) \longrightarrow (2,3,1) \longrightarrow (3,2,1)$$

3 STEPS.

$(1, 2, 3) \to (1, 3, 2) \to (3, 1, 2)$

$\to (2, 1, 3) \to (2, 3, 1) \to (3, 2, 1)$

5 STEPS.

We can never get from $(1,2,3)$ to $(3,2,1)$ using an even # of steps. So define

won't prove it.

$$\operatorname{sgn}(\sigma) = -1$$

Example S_3:

σ	$\operatorname{sgn}(\sigma)$
123	+1
231	+1
312	+1
132	−1
213	−1
321	−1

So 3×3 det is

$$\det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

$$= \boxed{\begin{aligned} &+ a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 \\ &- a_1 b_3 c_2 - a_2 b_1 c_3 - a_3 b_2 c_1 \end{aligned}}$$

In terms of tensors:

$$\det = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, \varepsilon_{\sigma(1)} \otimes \cdots \otimes \varepsilon_{\sigma(n)}$$

One needs to check that this actually satisfies 3 rules of a determinant function. OMITTED!

More Useful: Laplace Expansion.

Recursive definition of det.

Given $n \times n$ A let

$\hat{A}_{ij}$ = $(n-1)\times(n-1)$ matrix by delete ith row & jth col of A.

Expand along ith row. Fix i

$$\det(A) = \sum_j (-1)^{i+j} a_{ij} \det(\hat{A}_{ij})$$

Or expand along col j. Fix j

$$\det(A) = \sum_i (-1)^{i+j} a_{ij} \det(\hat{A}_{ij})$$

Example:

$$\det\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = \det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$= (-1)^{2+1} a_{21} \det(\hat{A}_{21})$$
$$+ (-1)^{2+2} a_{22} \det(\hat{A}_{22})$$
$$+ (-1)^{2+3} a_{23} \det(\hat{A}_{23})$$

$$= -a_2 \det\begin{pmatrix} b_1 & c_1 \\ b_3 & c_3 \end{pmatrix}$$
$$+ b_2 \det\begin{pmatrix} a_1 & c_1 \\ a_3 & c_3 \end{pmatrix}$$
$$- c_2 \det\begin{pmatrix} a_1 & b_1 \\ a_3 & b_3 \end{pmatrix}$$

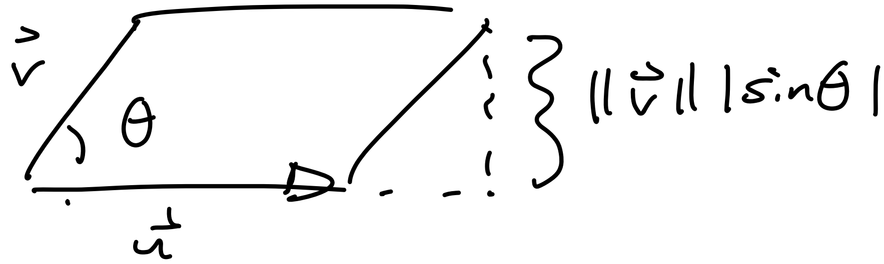
$$
\begin{aligned}
= & -a_2(b_1c_3 - b_3c_1) \\
& + b_2(a_1c_3 - a_3c_1) \\
& - c_2(a_1b_3 - a_3b_1)
\end{aligned}
$$

Optional Section on Cramer's Rule in the typed notes.

What is the determinant, really?

VOLUME.

Consider parallelogram



area $= \|\vec{u}\|\,\|\vec{v}\|\,|\sin\theta|$.

base × height

$$A = (\vec{u} \mid \vec{v})$$

$$A^T A = \begin{pmatrix} \|\vec{u}\|^2 & \vec{u}\circ\vec{v} \\ \vec{u}\circ\vec{v} & \|\vec{v}\|^2 \end{pmatrix}$$

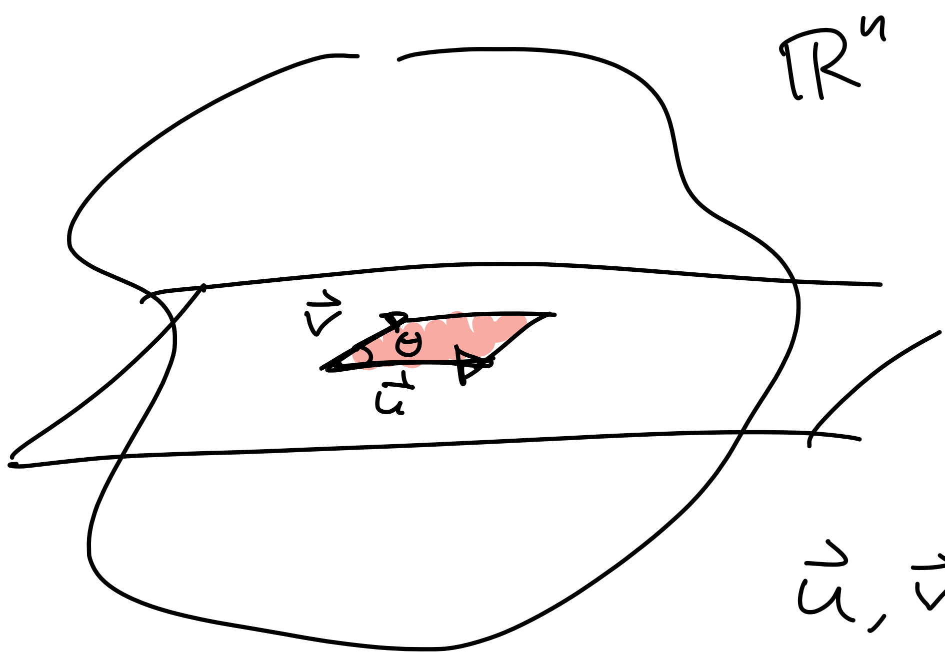
$$\det(A^T A) = \|\vec{u}\|^2\|\vec{v}\|^2 - (\vec{u}\circ\vec{v})^2$$

$$= \|\vec{u}\|^2\|\vec{v}\|^2 \sin^2\theta .$$

$$\sqrt{\det(A^T A)} = \|\vec{u}\|\|\vec{v}\| |\sin\theta|$$

$$= \underline{\underline{\text{area}}} .$$

2D Parallelogram in $\mathbb{R}^n$:



$\vec{u}, \vec{v} \in \mathbb{R}^n$

$$\sqrt{\det(A^TA)} = \|\vec{u}\|\|\vec{v}\| |\sin\theta|$$
$$= \text{area of parallelogram.}$$

even though A is $n \times 2$,

so $\det(A)$ <u>not defined</u>.

Special Case: Parallelogram in $\mathbb{R}^2$.

A is <u>square</u>, so $\det(A)$ exists,

$$\sqrt{\det(A^TA)} = \sqrt{\det(A^T)\det(A)}$$
$$= \sqrt{\det(A)\det(A)} = \sqrt{\det(A)^2}$$
$$= |\det(A)|.$$

General Theorem:

k-parallelogram gen by $\vec{a}_1, \dots, \vec{a}_k \in \mathbb{R}^n$.

Let $A = \begin{pmatrix} \vec{a}_1 & \cdots & \vec{a}_k \end{pmatrix}$ $n \times k$.

$$\mathrm{Vol}_k(A) = \sqrt{\det(A^TA)}$$

If $k=n$ then A is square:

$$\mathrm{Vol}_n(A) = |\det(A)|.$$

Proof? Several Steps.

- $\mathrm{Vol}_n(A) = |\det(A)|$ Hardest Part.

HW

- $|\det(A)| = \sqrt{\det(A^T A)}$
- $\det(A^T A)$ only depends on lengths & angles between cols of A.

?

- Hence for $n \times k$ A
  $$\mathrm{Vol}_k(A) = \sqrt{\det(A^T A)}$$

To prove $\mathrm{Vol}_n(A) = |\det(A)|$.

Actually $\det(A)$ = signed volume.

$\pm$

Need to show signed volume
satisfies 3 rules of determinants.