A basis of vector space V is a set $B = \{\vec{b}_1, \dots, \vec{b}_n\} \subseteq V$ such that every $\vec{v} \in V$ has a <u>unique</u> expression

$$\vec{v} = a_1 \vec{b}_1 + \dots + a_n \vec{b}_n$$

Steinitz Exchange (HW 1, Optional): If V has basis of size n, then any basis of V has size n & we say

$$\dim V = n.$$

e.g. $\dim \mathbb{R}^n = n$ because of the standard bases $\vec{e}_1, \dots, \vec{e}_n$.

A non-standard basis of $\mathbb{R}^3$ is

$$(1,1,1), (0,5,7), (0,0,-72).$$

Not every v.s. has a finite basis.

e.g. Polynomials $\mathbb{R}[x]$.

Every poly $f(x)$ has a unique expression

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

with $a_0, a_1, \ldots, a_n \in \mathbb{R}$.

We say $\{1, x, x^2, \ldots\}$ is a basis, but it's not finite.

KEY: Every $f(x) \in \mathbb{R}[x]$ is a finite linear combination of "basis vectors" $1, x, x^2, \ldots$.

Infinite linear combinations are called "formal power series"

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

convergence ?

Axioms : Inner product

- $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$
- $\langle a\vec{u}, \vec{v} \rangle = \langle \vec{u}, a\vec{v} \rangle = a \langle \vec{u}, \vec{v} \rangle$
- $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$.
- $\langle \vec{v}, \vec{v} \rangle \geq 0$
- $\langle \vec{v}, \vec{v} \rangle = 0 \iff \vec{v} = \vec{0}$.

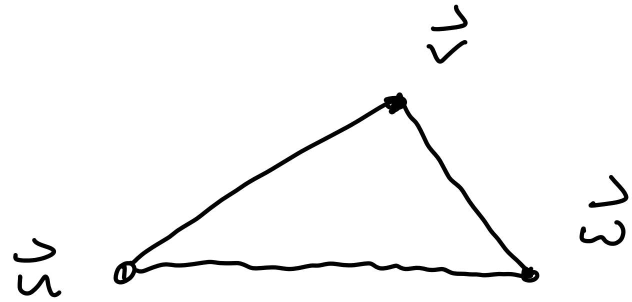
Normed linear space:

- $\|\vec{v}\| \geq 0$
- $\|\vec{v}\| = 0 \iff \vec{v} = \vec{0}$
- $\|a\vec{v}\| = |a| \, \|\vec{v}\|$
- $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$

Metric Linear Space :

- $dist(\vec{u}, \vec{v}) \geq 0$
- $dist(\vec{u}, \vec{v}) = 0 \iff \vec{u} = \vec{v}$.
- $dist(\vec{u}, \vec{v}) = dist(\vec{v}, \vec{u})$
- $dist(\vec{u}, \vec{v}) \leq dist(\vec{u}, \vec{w}) + dist(\vec{w}, \vec{v})$

Vague Picture:



Theorem:

Inner Product $\rightsquigarrow$ Norm $\longrightarrow$ Metric.

(HW 1.1-2)

$$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$$

$$\text{dist}(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|.$$

Using a metric we can say when ∞ linear combinations <u>converge</u>.

Example: $L^2[0,1]$

$$\langle f(x), g(x) \rangle = \int_0^1 f(x) g(x)\, dx$$

$$\| f(x) - g(x) \|^2 = \langle f(x) - g(x), f(x) - g(x) \rangle$$

$$= \int_0^1 (f(x) - g(x))^2 \, dx$$

HW 1.4.

$$s_n(x) = \sqrt{2} \sin(2\pi n x)$$
$$c_n(x) = \sqrt{2} \cos(2\pi n x).$$

$$\langle s_m(x), c_n(x) \rangle = 0 \quad \forall m, n$$

$$\langle s_m(x), s_n(x) \rangle = \delta_{mn}$$

$$\langle c_m(x), c_n(x) \rangle = \delta_{mn}$$

$$\langle 1, s_n(x) \rangle = 0$$

$$\langle 1, c_n(x) \rangle = 0.$$

Vectors $\vec{b}_1, \ldots, \vec{b}_n$ called orthogonal if

$$\langle \vec{b}_i, \vec{b}_j \rangle = 0 \quad \text{for } i \neq j$$

[In Euclidean space
orthog. $\iff$ perpendicular.]

Say orthonormal if

$$\langle \vec{b}_i, \vec{b}_i \rangle = 1.$$

$$\|\vec{b}_i\| = \sqrt{1} = 1.$$

ALWAYS prefer to work with O.N. sets of vectors.

$$\vec{v} = a_1 \vec{b}_1 + \cdots + a_n \vec{b}_n,$$

Easy to find coeffs:

$$\langle \vec{v}, \vec{b}_i \rangle = \langle a_1 \vec{b}_1 + \cdots + a_n \vec{b}_n, \vec{b}_i \rangle$$

$$= a_1 \langle \vec{b}_1, \vec{b}_i \rangle + \cdots + a_n \langle \vec{b}_n, \vec{b}_i \rangle$$

$$= a_1 0 + \cdots + a_i 1 + \cdots + a_n 0$$

$$= a_i$$

$$\boxed{a_i = \langle \vec{v}, \vec{b}_i \rangle}$$

Example: Fourier Series

$$1, \quad s_n(x) = \sqrt{2} \sin(2\pi n x),$$
$$c_n(x) = \sqrt{2} \cos(2\pi n x).$$

Assume we can write

$$f(x) = a_0 1 + \sum_{n \geq 1} a_n s_n(x)$$
$$+ \sum_{n \geq 1} b_n c_n(x)$$

[Convergence in the sense of the standard inner product.]

FOURIER'S TRICK:

"Easy" to find the coefficients.

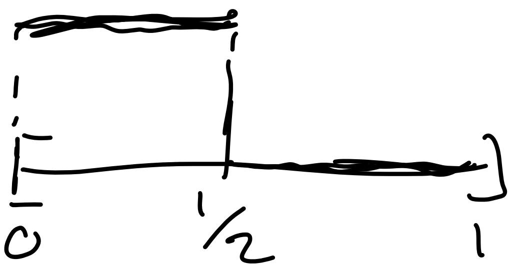
$$a_0 = \langle f(x), 1 \rangle = \int f(x)\, dx$$
$$a_n = \langle f(x), s_n(x) \rangle = \int f(x) s_n(x)\, dx$$
$$b_n = \langle f(x), c_n(x) \rangle = \int f(x) c_n(x)\, dx.$$

You can TRY this with any function $f : [0,1] \to \mathbb{R}$

and hope that the resulting infinite series converges.

e.g. Square Wave:

$$f(x) = \begin{cases} 1 & 0 \le x < 1/2, \\ 0 & 1/2 \le x \le 1. \end{cases}$$



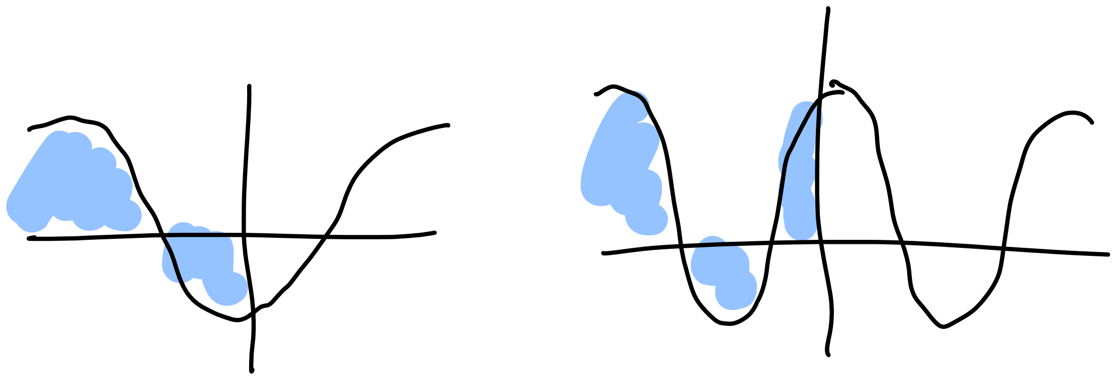
$$a_0 = \langle f(x), 1 \rangle = \int_0^1 f(x)\,dx = 1/2$$

$$b_n = \langle f(x), c_n(x) \rangle$$

$$= \int_0^1 f(x)\, c_n(x)\, dx$$

$$= \int_0^{1/2} \sqrt{2} \cos(2\pi n x)\, dx$$

$$= 0 \quad \text{for any } n.$$



$$a_n = \langle f(x), s_n(x) \rangle$$

$$= \int_0^{1/2} \sqrt{2} \sin(2\pi n x)\, dx$$

$$= \sqrt{2} \left[-\frac{\cos(2\pi n x)}{2\pi n} \right]_0^{1/2}$$

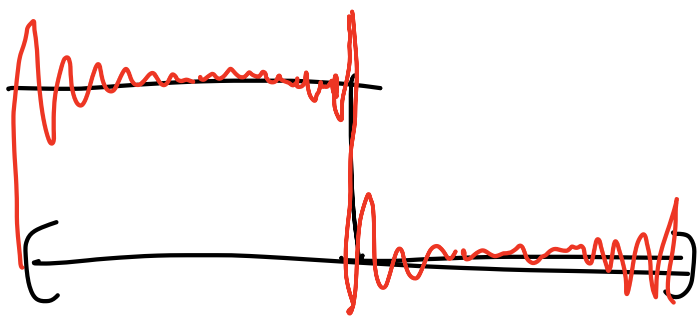
$$= \frac{\sqrt{2}}{2\pi n} \left(-\cos(\pi n) + \cos(0) \right)$$

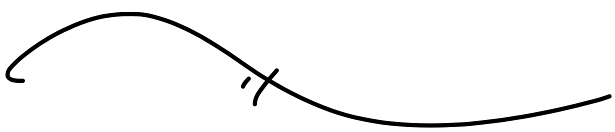
$$= \frac{\sqrt{2}}{2\pi n} \left(-(-1)^n + 1 \right)$$

$$= \begin{cases} 0 & n \text{ even} \\ \sqrt{2}/\pi n & n \text{ odd.} \end{cases}$$

Conclusion:

$$f(x) = \frac{1}{2} + \frac{\sqrt{2}}{\pi} s_1(x) + \frac{\sqrt{2}}{3\pi} s_3(x) + \frac{\sqrt{2}}{5\pi} s_5(x) + \cdots$$





Parseval's Identity:

O.N. set $\{\vec{b}_1, \ldots, \vec{b}_n\}$.

$$\vec{v} = a_1\vec{b}_1 + \cdots + a_n\vec{b}_n$$

$$\|\vec{v}\|^2 = a_1^2 + a_2^2 + \cdots + a_n^2 \text{.} \quad \text{(HW)}$$

For Fourier Series

$$f(x) = a_0 + \sum a_n s_n(x) + \sum b_n c_n(x)$$

$$\int_0^1 f(x)^2\,dx = a_0^2 + \sum a_n^2 + \sum b_n^2 .$$

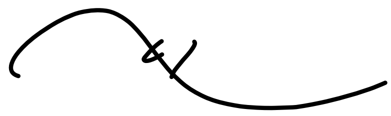
Ex:

$$\text{square wave} = \frac{1}{2} + \frac{\sqrt{2}}{\pi} s_1(x) + \frac{\sqrt{2}}{3\pi} s_3(x) + \frac{\sqrt{2}}{5\pi} s_5(x) + \cdots$$

$$\overset{1/2}{\cancel{\int_0^1 f(x)^2\,dx}} = \frac{1}{4} + \frac{2}{\pi^2} + \frac{2}{9\pi^2} + \frac{2}{25\pi^2}$$

$$\frac{1}{4} = \frac{2}{\pi^2}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$



Related:

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = ?$$

Bernoulli:

Euler: $\pi^2/6$.

Now these identities fall out magically from Fourier series.