Concept of a basis.

Let $V/\mathbb{R}$ be vector space

Let $B \subseteq V$ be subset.

Say $B = \{\vec{b}_1, \dots, \vec{b}_n\}$.

- B <u>spanning</u>:

for all there exist

$\forall \vec{v} \in V$, $\exists a_1, \dots, a_n \mathbb{R}$,

$$\vec{v} = a_1 \vec{b}_1 + \dots + a_n \vec{b}_n$$

- B independent

(linearly independent over $\mathbb{R}$)

if $a_1 \vec{b}_1 + \dots + a_n \vec{b}_n = \vec{0}$

$$\Longrightarrow a_1 = 0, a_2 = 0, \dots, a_n = 0.$$

i.e. no nontrivial linear relation among the $\vec{b}_i$.

Say $\vec{b}_3 = 2\vec{b}_2 + \vec{b}_1$.

<u>Not</u> independent.

Technically:

$$1\vec{b}_1 + 2\vec{b}_2 - 1\vec{b}_3 = \vec{0}$$

coeffs not all zero.

Also implies *unique* expressions.

Say

$$a_1\vec{b}_1 + \cdots + a_n\vec{b}_n = c_1\vec{b}_1 + \cdots + c_n\vec{b}_n$$

$$(a_1 - c_1)\vec{b}_1 + \cdots + (a_n - c_n)\vec{b}_n = \vec{0}$$

If $\vec{b}_i$ are independent

then $a_i - c_i = 0 \quad \forall i$

$a_i = c_i$.

- Alternate: Say B independent if

$\forall \vec{v} \in V$, $\exists$ at most one choice (possibly zero)

of scalars $a_1, \dots, a_n$ s.t.

$$\vec{v} = a_1\vec{b}_1 + \cdots + a_n\vec{b}_n.$$

∘ Basis = Independent + Spanning.

i.e. $\forall \vec{v} \in V$, $\exists$ unique scalars $a_1, \dots, a_n$ s.t.

$$\vec{v} = a_1\vec{b}_1 + \cdots + a_n\vec{b}_n$$

$$\text{“} \vec{v} = (a_1, a_2, \dots, a_n)_B \text{”}$$

Important Theorem: If V has a finite basis, say of size n, then every basis has size n. We define

$$\dim V = n = \text{the size of any basis.}$$

e.g. $\dim \mathbb{R}^n = n$ ✓ :)

Proof (Steinitz Exchange, 1913).

V = vector space

$I \subseteq V$ independent (finite)

$S \subseteq V$ spanning set (Finite).

We will show $\# I \leq \# S$.

[Having shown this, let B_1, B_2 be two bases. Then

$$\underbrace{\# B_1}_{\text{ind.}} \leq \underbrace{\# B_2}_{\text{span.}} \quad \& \quad \underbrace{\# B_2}_{\text{ind.}} \leq \underbrace{\# B_1}_{\text{span.}}$$

Hence $\# B_1 = \# B_2$. ✓]

Steinitz: Let $k \leq \# I$.

Then for any $\vec{u}_1, \ldots, \vec{u}_k \in I$ we can exchange k vectors from S to define (say $\# S = n$).

$$\{ \underbrace{\vec{u}_1, \ldots, \vec{u}_k, \vec{s}_1, \vec{s}_2, \ldots, \vec{s}_{n-k}}_{\text{such that this is still a spanning set.}} \}$$

e.g. Exchange one element of S.

Let $S = \{ \vec{s}_1, \ldots, \vec{s}_n \}$.

Pick any $\vec{u} \in I$.

Since S spans, can write

$$\vec{u} = a_1 \vec{s}_1 + \cdots + a_n \vec{s}_n$$

But not all $a_i = 0$ because $\vec{u} \neq \vec{0}$.

Say e.g. $a_1 \neq 0$. Then

$$a_1 \vec{s}_1 = \vec{u} - a_2 \vec{s}_2 - \cdots - a_n \vec{s}_n$$

$$\vec{s}_1 = \frac{1}{a_1}\vec{u} - \frac{a_2}{a_1}\vec{s}_2 - \cdots - \frac{a_n}{a_1}\vec{s}_n$$

This tells me that

$$\{\vec{u}, \vec{s}_2, \ldots, \vec{s}_n\}$$

is a spanning set. Because for any $\vec{v}$ we can write

$$\vec{v} = c_1 \vec{s}_1 + c_2 \vec{s}_2 + \cdots + c_n \vec{s}_n$$

$$= c_1 \underbrace{\left(\frac{1}{a_1}\vec{u} - \cdots\right)}_{\text{expand}} + c_2 \vec{s}_2 + \cdots + c_n \vec{s}_n$$

$$\vec{v} = ?\,\vec{u} + ?\,\vec{s}_2 + ?\,\vec{s}_3 + \cdots + ?\,\vec{s}_n$$

so they span.

Continue:

$$\{\vec{u}_1, \vec{u}_2, \vec{s}_3, \vec{s}_4, \dots, \vec{s}_n\} \text{ spans.}$$

$$\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{s}_4, \dots, \vec{s}_n\}.$$

If $\# I \geq \# S$

$\vdots$

$$\{\vec{u}_1, \dots, \vec{u}_n\} \text{ spanning.}$$

But then since $\vec{u}_1, \dots, \vec{u}_n$ span we have nontrivial expression

$$\vec{u}_{n+1} = a_1\vec{u}_1 + \dots + a_n\vec{u}_n$$

Contradicts fact that $\vec{u}_i$ are independent.

QED.

Example: $L^2[0,1]$ does not have a <u>finite</u> basis. But we can still do something.

Let $V/\mathbb{R}$ have infinite dimension.

Given $B = \{\vec{b}_1, \vec{b}_2, \dots \}$.

Say B independent if there are no finite relations among the $\vec{b}_i$.

Say B spanning if every $\vec{v} \in V$ has a convergent expression

$$\vec{v} = \underbrace{a_1 \vec{b}_1 + a_2 \vec{b}_2 + \cdots}_{\text{sum converges.}}$$

$$\| \vec{v} - (a_1 \vec{b}_1 + \cdots + a_k \vec{b}_k) \| \longrightarrow 0$$

as $k \longrightarrow \infty$.

In $L^2[0,1]$ we have

$$\langle f, g \rangle = \int_0^1 f(x) g(x)\, dx$$

so we can define distance:

$$\mathrm{dist}(f,g) = \| f - g \|$$

$$= \sqrt{\int_0^1 (f(x)-g(x))^2\,dx}$$

BIG THEOREM:

$1, \sqrt{2}\sin(2\pi kx), \sqrt{2}\cos(2\pi kx)$

$k = 1, 2, 3, 4, \ldots$

is a basis for $L^2[0,1]$.

Independence is easier. One checks that any two of these functions $b_i(x)$ & $b_j(x)$ have

$$\langle b_i(x), b_j(x) \rangle = \begin{cases} 0 & i \neq j, \\ 1 & i = j. \end{cases}$$

$$= \delta_{ij}$$

Suppose we had some finite linear relation:

$$a_1 b_1(x) + a_2 b_2(x) + \cdots + a_k b_k(x) = 0$$

Take inner product with $b_i(x)$ on both sides:

$$\langle 0, b_i(x) \rangle = \sum_{j=1}^{k} a_j \overset{\delta_{ij}}{\cancel{\langle b_j(x), b_i(x) \rangle}}$$

$$= \cancel{a_1 \cdot 0} + \cdots + a_i \cdot 1 + \cdots \cancel{a_k \cdot 0}$$

$$\boxed{0 = a_i}$$

So every coefficient is zero ✓.

SPANNING: One of the deepest theorems in mathematics.