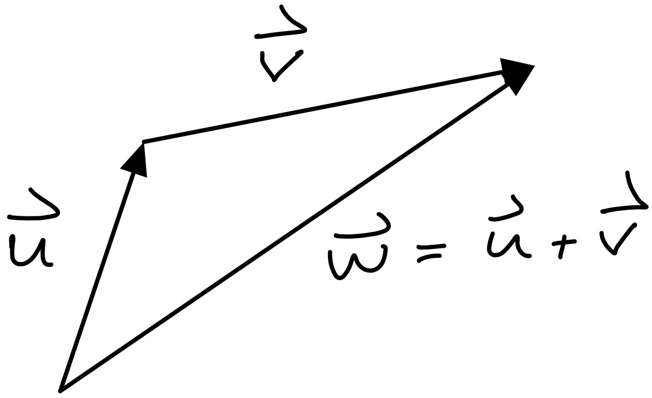
Continue review of Euclidean Space.

Addition of arrows:



It makes sense to write

$$\vec{u} + \vec{v} = \vec{w}$$
$$\vec{u} = \text{“}\vec{w} - \vec{v}\text{”}$$
$$\vec{v} = \text{“}\vec{w} - \vec{u}\text{”}$$

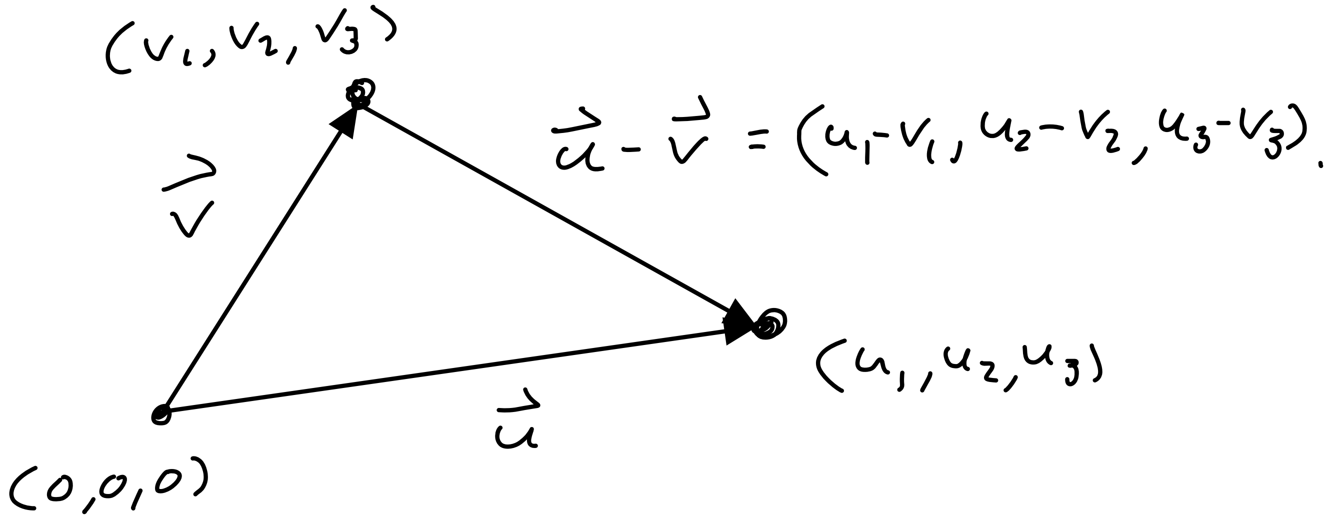
In terms of coordinates

$$\vec{u} = (u_1, \dots, u_n)$$
$$\vec{v} = (v_1, \dots, v_n)$$

$$\vec{u} - \vec{v} = (u_1 - v_1, \dots, u_n - v_n)$$

MNEMONIC:

arrow in coordinates = head minus tail



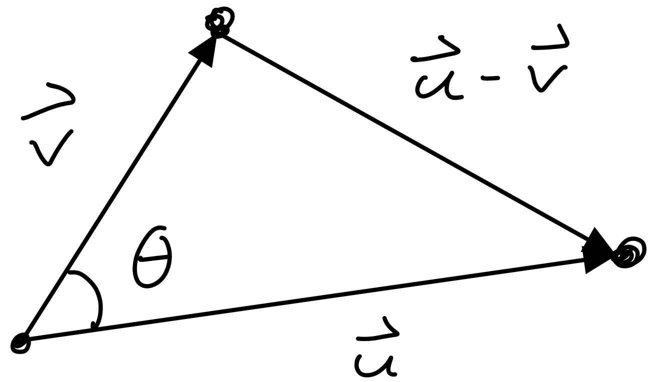
The distance between points is the length of their difference:

$$\text{dist}(\vec{u}, \vec{v}) = \| \vec{u} - \vec{v} \|$$

$$= \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}$$

Dot Product

We can discover the dot product by considering the triangle:



Let θ be angle between $\vec{u}$ & $\vec{v}$ placed tail-to-tail.

The Law of Cosines:

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta$$

On the other hand, pure algebra says that

$$\|\vec{u}-\vec{v}\|^2 = (u_1-v_1)^2 + \cdots + (u_n-v_n)^2$$

$$= (u_1^2 - 2u_1v_1 + v_1^2) + \cdots + (u_n^2 - 2u_nv_n + v_n^2)$$

$$= (u_1^2 + \cdots + u_n^2) + (v_1^2 + \cdots + v_n^2)$$

$$- 2(u_1v_1 + \cdots + u_nv_n)$$

$$= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2(u_1v_1 + \cdots + u_nv_n).$$

Compare expressions to get

$$\|\vec{u}\|\|\vec{v}\| \cos\theta = \underbrace{u_1v_1 + \cdots + u_nv_n}_{\text{Define } \vec{u} \bullet \vec{v}}.$$

dot product.

Observe:

$$\vec{u} \bullet \vec{u} = u_1u_1 + u_2u_2 + \cdots + u_nu_n$$
$$= u_1^2 + u_2^2 + \cdots + u_n^2$$
$$= \|\vec{u}\|^2.$$

So we can express the angle θ purely in terms of dot products:

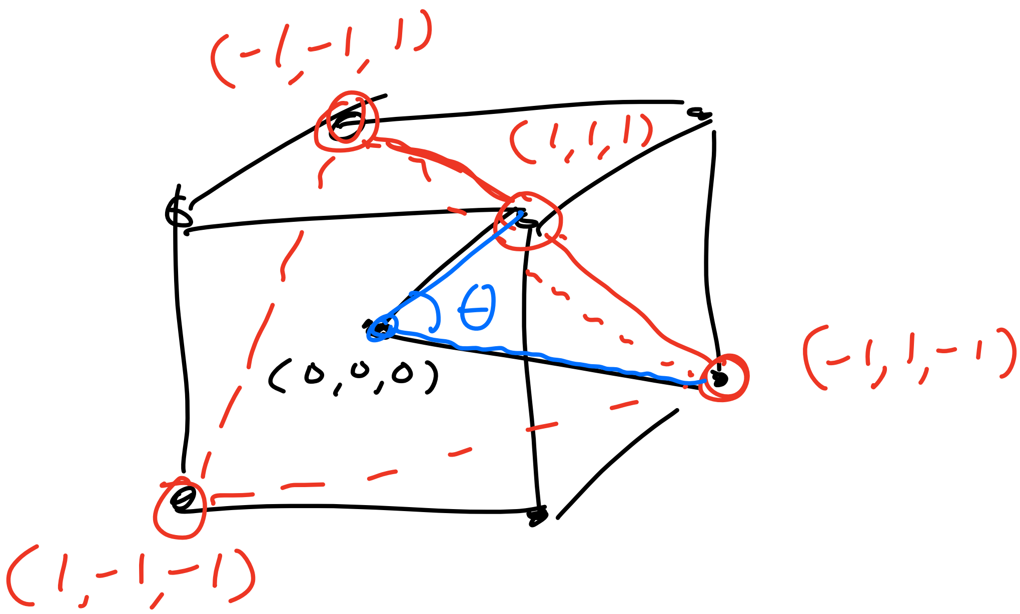
$$\|\vec{u}\|\|\vec{v}\| \cos\theta = \vec{u} \bullet \vec{v}$$

$$\cos\theta = \frac{\vec{u} \bullet \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$$

$$= \frac{\vec{u} \circ \vec{v}}{\sqrt{\vec{u} \circ \vec{u}}\ \sqrt{\vec{v} \circ \vec{v}}}$$

Example: The tetrahedral angle.

points $(\pm 1, \pm 1, \pm 1)$ are the vertices of a cube:



Let $\vec{u} = (1,1,1)$

$\vec{v} = (-1,1,-1)$

$$\cos\theta = \vec{u} \circ \vec{v} \Big/ \sqrt{\vec{u} \circ \vec{u}}\ \sqrt{\vec{v} \circ \vec{v}}$$

$$= -1 \Big/ \sqrt{3}\sqrt{3} = -1/3$$

$\theta = \arccos(-1/3)$.

$\approx 109.47°$

General Vector Space (over $\mathbb{R}$)

(later over $\mathbb{C}$)

A <u>vector space</u> (over $\mathbb{R}$) is a set V (of "vectors") with two operations:

$\vec{u}, \vec{v} \in V \implies \vec{u} + \vec{v} \in V$

$a \in \mathbb{R}, \vec{v} \in V \implies a\vec{v} \in V$

satisfying following rules:

$\vec{u} + \vec{v} = \vec{v} + \vec{u}$

$\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$

$\exists\ \vec{0} \in V,\ \vec{0} + \vec{v} = \vec{v} \quad \forall \vec{v} \in V$

$\forall \vec{v} \in V, \exists \vec{u} \in V, \vec{v} + \vec{u} = \vec{0}$

[Think $\vec{u} = -\vec{v}$. It follows

that $\vec{w} + \vec{v} = \vec{v}$

$$\Rightarrow \quad \vec{w} + \cancel{\vec{v} + \vec{u}} = \cancel{\vec{v} + \vec{u}}$$

$$\vec{w} + \vec{0} = \vec{0}$$

$$\vec{w} = \vec{0}. \quad]$$

$$a(b\vec{v}) = (ab)\vec{v}$$

$$a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$$

$$(a+b)\vec{v} = a\vec{v} + b\vec{v}.$$

[$\exists$ = "there exists"

$\forall$ = "for all"]

Direct consequences:

$$0\vec{v} = \vec{0}$$

Proof: $0 + 0 = 0$

For any $\vec{v} \in V$ have

$$(0+0)\vec{v} = 0\vec{v}$$
$$0\vec{v} + \cancel{0\vec{v}} = \cancel{0\vec{v}}$$
$$0\vec{v} = \vec{0} \quad \checkmark$$

Furthermore, we call V an inner product space if we have a function

$$\vec{u},\vec{v} \in V \implies \langle \vec{u},\vec{v} \rangle \in \mathbb{R}$$

$\langle \vec{u} | \vec{v} \rangle$ in Physics

satisfying

$$\langle \vec{u}, a\vec{v} \rangle = \langle a\vec{u}, \vec{v} \rangle = a\langle \vec{u},\vec{v} \rangle.$$
$$\langle \vec{u},\vec{v} \rangle = \langle \vec{v},\vec{u} \rangle$$
$$\langle \vec{u}, \vec{v}+\vec{w} \rangle = \langle \vec{u},\vec{v} \rangle + \langle \vec{u},\vec{w} \rangle$$
$$\langle \vec{u}+\vec{v}, \vec{w} \rangle = \langle \vec{u},\vec{w} \rangle + \langle \vec{v},\vec{w} \rangle$$
$$\langle \vec{v},\vec{v} \rangle \geq 0 \quad \forall \vec{v} \in V$$
$$\langle \vec{v},\vec{v} \rangle = 0 \iff \vec{v} = \vec{0}$$

[Over $\mathbb{C}$:

$$\langle \vec{u}, \vec{v} \rangle = \overline{\langle \vec{v}, \vec{u} \rangle}$$

$$\langle a\vec{u}, \vec{v} \rangle = a \langle \vec{u}, \vec{v} \rangle$$

$$\langle \vec{u}, a\vec{v} \rangle = \bar{a} \langle \vec{u}, \vec{v} \rangle.$$

Called a "Hermitian inner product".]

Example: $\mathbb{R}^n$ is an inner product space (over $\mathbb{R}$) with operations

$$(u_1, \ldots, u_n) + (v_1, \ldots, v_n) = (u_1 + v_1, \ldots, u_n + v_n)$$

$$a(v_1, \ldots, v_n) = (av_1, \ldots, av_n)$$

$$\langle (u_1, \ldots, u_n), (v_1, \ldots, v_n) \rangle = u_1 v_1 + \cdots + u_n v_n.$$

So why bother with the abstract definition??

Example. Let $L^2[0,1]$ be the

set of functions $[0,1] \to \mathbb{R}$ such that the integral

$$\int_0^1 f(x)^2\,dx \text{ exists.}$$

Define addition & scalar mult of functions

$$(f+g)(x) = f(x) + g(x)$$

$$(af)(x) = af(x)$$

Define an inner product

$$\langle f(x), g(x) \rangle = \int_0^1 f(x)g(x)\,dx$$

$$\int_0^1 f(x)\overline{g(x)}\,dx$$

in Physics

Then $L^2[0,1]$ is an inner product space.