Math 510 **Fall 2022**
Notes for Homework 5 **Drew Armstrong**

Contents

1 Eigenvalues and Eigenvectors

1.1 A Motivating Example

In order to motivate the concepts of eigenvalues and eigenvectors I will develop one specific example in detail. Some of the steps will seem miraculous, and only make sense later when we discuss the general theory.

Our example comes from the theory of "Markov chains". Consider the following matrix:

$$A = \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}.$$

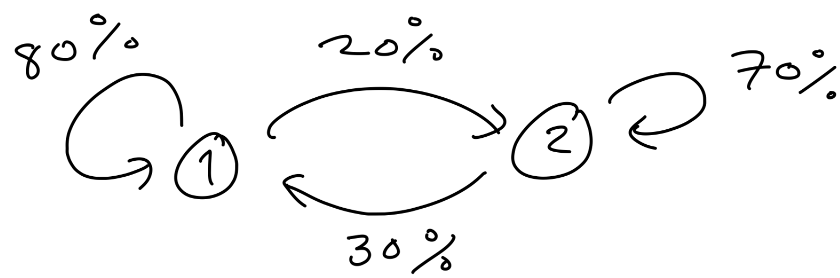
This matrix has the special property that each of its columns sums to 1. In matrix notation:[1]

$$\begin{pmatrix}1 & 1\end{pmatrix} A = \begin{pmatrix}1 & 1\end{pmatrix} \begin{pmatrix}.8 & .3 \\ .2 & .7\end{pmatrix} = \begin{pmatrix}.8+.2 & .3+.7\end{pmatrix} = \begin{pmatrix}1 & 1\end{pmatrix}.$$

By induction, this implies that every power of A has columns that sum to 1:

$$\begin{aligned}
\begin{pmatrix}1 & 1\end{pmatrix} A^n &= \begin{pmatrix}1 & 1\end{pmatrix} (AA^{n-1}) \\
&= (\begin{pmatrix}1 & 1\end{pmatrix} A)A^{n-1} \\
&= \begin{pmatrix}1 & 1\end{pmatrix} A^{n-1} \\
&\ \vdots \\
&= \begin{pmatrix}1 & 1\end{pmatrix}.
\end{aligned}$$

Such a matrix is called a *Markov matrix* or a *stochastic matrix*. We can interpret the matrix entries as probabilities. Suppose that a certain particle can be in one of two states. At each discrete time step, the particle can change states, according to the following probabilities:



That is, if the particle is currently in state 1 then it has an 80% chance to stay in state 1 and a 20% chance to transition to state 2. If the particle is in state 2 then it has a 70% chance to stay and a 30% chance to stay. This is why the columns of A must sum to 1.

Now let's consider the first few powers of A:

$$\begin{aligned}
A &= \begin{pmatrix}.8 & .3 \\ .2 & .7\end{pmatrix}, \\
A^2 &= \begin{pmatrix}.7 & .45 \\ .3 & .55\end{pmatrix}, \\
A^3 &= \begin{pmatrix}.65 & .525 \\ .35 & .475\end{pmatrix}, \\
&\ \vdots \\
A^{10} &= \begin{pmatrix}0.600390625 & 0.5994140625 \\ 0.399609375 & 0.4005859375\end{pmatrix}.
\end{aligned}$$

[1] Jargon: Later we will say that $\begin{pmatrix}1 & 1\end{pmatrix}$ is a "left eigenvector" of A with "eigenvalue" 1.

Do you see any pattern here? It seems likely that

$$A^n \to \begin{pmatrix} .6 & .6 \\ .4 & .4 \end{pmatrix} \text{ as } n \to \infty,$$

but the entries of the matrices look complicated. Nevertheless, at the end of this section we will obtained exact formulas for the entries of each power A^n.

We have shown that each column of each power A^n sums to 1. This fact has a probabilistic interpretation. Let p_k and q_k be the probabilities that the particle is in state 1 or 2, respectively, after k seconds, and let p_0, q_0 denote the initial probabilities. Then I claim that[2]

$$\boxed{\mathbf{p}_n = \begin{pmatrix} p_n \\ q_n \end{pmatrix} = A \begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix} = A\mathbf{p}_{n-1}.}$$

To prove this we use the *law of total probability* (which I won't explain here). Given any two events S and T, we have the following identities:

$$\begin{aligned} P(S) &= P(T)P(S|T) + P(T')P(S|T'), \\ P(S') &= P(T)P(S'|T) + P(T')P(S'|T'). \end{aligned}$$

Let S_n be the event that "the particle is in state 1 after n seconds", so that

$$p_n = P(S_n) \text{ and } q_n = P(S_n').$$

The transition matrix A tells us that

$$\begin{aligned} P(S_n|S_{n-1}) &= .8, \\ P(S_n'|S_{n-1}) &= .2, \\ P(S_n|S_{n-1}') &= .3, \\ P(S_n'|S_{n-1}') &= .7, \end{aligned}$$

which are independent of n. Hence we have

$$\begin{aligned} p_n &= P(S_n) \\ &= P(S_{n-1})P(S_n|S_{n-1}) + P(S_{n-1}')P(S_n|S_{n-1}') \\ &= p_{n-1}(.8) + q_{n-1}(.3) \end{aligned}$$

and

$$\begin{aligned} q_n &= P(S_n') \\ &= P(S_{n-1})P(S_n'|S_{n-1}) + P(S_{n-1}')P(S_n'|S_{n-1}') \\ &= p_{n-1}(.2) + q_{n-1}(.7), \end{aligned}$$

[2] Alternatively, suppose that we have an ensemble of particles and let x_n, y_n denote the **expected number** of particles in each state. Then the same theory will hold.

as desired. But enough about probability.

Given the initial distribution $\mathbf{p}_0 = (p_0, q_0)$, the distribution after n seconds is given by

$$\begin{aligned}
\mathbf{p}_n &= A\mathbf{p}_{n-1}, \\
&= AA\mathbf{p}_{n-2} \\
&\;\vdots \\
&= AA\cdots A\mathbf{p}_0 \\
&= A^n\mathbf{p}_0.
\end{aligned}$$

Our goal is to find explicit formulas for p_n and q_n in terms of p_0 and q_0.

Now comes the key trick. We have the following mysterious identitites:

$$A\begin{pmatrix}3\\2\end{pmatrix} = \begin{pmatrix}3\\2\end{pmatrix} \quad \text{and} \quad A\begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}. \tag{$*$}$$

Jargon: We say that $(3, 2)$ and $(1, -1)$ are "eigenvectors" of A with corresponding "eigenvalues" 1 and 1/2. More generally, if $A\mathbf{x} = \lambda\mathbf{x}$ for some vector $\mathbf{x}$ and scalar λ, then the action of A^n on $\mathbf{x}$ is easy to compute:

$$\begin{aligned}
A^n\mathbf{x} &= (A^{n-1}A)\mathbf{x} \\
&= A^{n-1}(A\mathbf{x}) \\
&= A^{n-1}(\lambda\mathbf{x}) \\
&= \lambda A^{n-1}\mathbf{x} \\
&\;\vdots \\
&= \lambda^n\mathbf{x}.
\end{aligned}$$

Once we know $(*)$, the rest of the solution is straightforward. First we want to express our initial condition $\mathbf{p}_0$ as a linear combination of eigenvectors. In other words, we want to find a and b such that

$$\begin{aligned}
\mathbf{p}_0 &= a\begin{pmatrix}3\\2\end{pmatrix} + b\begin{pmatrix}1\\-1\end{pmatrix} \\
&= \begin{pmatrix}3 & 1\\2 & -1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}.
\end{aligned}$$

Since the two eigenvectors are not parallel, the matrix of eigenvectors is invertible, hence

$$\begin{aligned}
\begin{pmatrix}a\\b\end{pmatrix} &= \begin{pmatrix}3 & 1\\2 & -1\end{pmatrix}^{-1}\begin{pmatrix}p_0\\q_0\end{pmatrix} \\
&= -\frac{1}{5}\begin{pmatrix}-1 & -1\\-2 & 3\end{pmatrix}\begin{pmatrix}p_0\\q_0\end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{5}\begin{pmatrix} -p_0 - q_0 \\ -2p_0 + 3q_0 \end{pmatrix} \\
&= \begin{pmatrix} 1/5 \\ p_0 - 3/5 \end{pmatrix}. && p_0 + q_0 = 1
\end{aligned}
$$

Then we obtain the solution:

$$
\begin{aligned}
\mathbf{p}_0 &= \frac{1}{5}\begin{pmatrix} 3 \\ 2 \end{pmatrix} + \left(p_0 - \frac{3}{5}\right)\begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
A^n\mathbf{p}_0 &= \frac{1}{5}A^n\begin{pmatrix} 3 \\ 2 \end{pmatrix} + \left(p_0 - \frac{3}{5}\right)A^n\begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
\mathbf{p}_n &= \frac{1}{5}\begin{pmatrix} 3 \\ 2 \end{pmatrix} + \left(p_0 - \frac{3}{5}\right)\left(\frac{1}{2}\right)^n\begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
\begin{pmatrix} p_n \\ q_n \end{pmatrix} &= \begin{pmatrix} 3/5 + (p_0 - 3/5)/2^n \\ 2/5 - (p_0 - 3/5)/2^n \end{pmatrix}.
\end{aligned}
$$

As $n \to \infty$ we observe that $\mathbf{p}_n \to (3/5, 2/5)$, regardless of the initial probabilities p_0 and q_0. The fact that the initial condition is irrelevant is sometimes called the "ergodic property" (or the "mixing property").

But we can do more. Suppose that $A\mathbf{x}_1 = \lambda_1\mathbf{x}_1$ and $A\mathbf{x}_2 = \lambda_2\mathbf{x}_2$ for some eigenvectors $\mathbf{x}_1, \mathbf{x}_2$ and eigenvalues λ_1, λ_2. We can express both of these conditions simultaneously by forming the matrices

$$
X = \left(\ \mathbf{x}_1 \mid \mathbf{x}_2\ \right) \quad \text{and} \quad \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.
$$

Then we have

$$
\begin{aligned}
AX &= A\left(\ \mathbf{x}_1 \mid \mathbf{x}_2\ \right) \\
&= \left(\ A\mathbf{x}_1 \mid A\mathbf{x}_2\ \right) \\
&= \left(\ \lambda_1\mathbf{x}_1 \mid \lambda_2\mathbf{x}_2\ \right) \\
&= \left(\ \mathbf{x}_1 \mid \mathbf{x}_2\ \right)\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \\
&= X\Lambda.
\end{aligned}
$$

This equation holds even when A is $n \times n$ and X is $n \times 2$. If A is 2×2 and if the vectors $\mathbf{x}_1, \mathbf{x}_2$ are not parallel, then the matrix X is square and invertible, hence

$$
\begin{aligned}
AX &= X\Lambda \\
A &= X\Lambda X^{-1}.
\end{aligned}
$$

In this case, we say that we have "diagonalized" the matrix A. In our case, we have

$$
\begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}^{-1}.
$$

The powers of A behave well with respect to this factorization. This follows from two key properties. First, the powers of a diagonal matrix are easy to compute:

$$\Lambda^n = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}^n = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}.$$

Second, there is a miraculous cancellation in the powers of $X\Lambda X^{-1}$:

$$\begin{aligned} A^n &= (X\Lambda X^{-1})^n \\ &= (X\Lambda X^{-1})(X\Lambda X^{-1})\cdots(X\Lambda X^{-1}) \\ &= X\Lambda(X^{-1}X)\Lambda(X^{-1}X)\cdots(X^{-1}X)\Lambda X^{-1} \\ &= X\Lambda\Lambda\cdots\Lambda X^{-1} \\ &= X\Lambda^n X^{-1}. \end{aligned}$$

Putting these together gives us explicit formulas for the entries of A^n:

$$\begin{aligned} \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}^n &= \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}^n \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1^n & 0 \\ 0 & (1/2)^n \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1^n & 0 \\ 0 & (1/2)^n \end{pmatrix} \left(-\frac{1}{5}\right) \begin{pmatrix} -1 & -1 \\ -2 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1^n & 0 \\ 0 & (1/2)^n \end{pmatrix} \left(-\frac{1}{5}\right) \begin{pmatrix} -1 & -1 \\ -2 & 3 \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (1/2)^n \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2/2^n & -3/2^n \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} 3 + 2/2^n & 3 - 3/2^n \\ 2 - 2/2^n & 2 + 3/2^n \end{pmatrix}. \end{aligned}$$

These exact formulas would be very difficult to obtain without the trick of eigenvalues and eigenvectors. By letting n go to infinity, we confirm our experimental observation that

$$A^n \to \frac{1}{5} \begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} .6 & .6 \\ .4 & .4 \end{pmatrix} \quad \text{as } n \to \infty.$$

Finally, let me mention an alternative expression for A^n that is often more useful. For any vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2$ and scalars λ_1, λ_2, one can check that

$$\left(\begin{array}{c|c} \mathbf{x}_1 & \mathbf{x}_2 \end{array} \right) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \left(\begin{array}{c} \mathbf{y}_1^T \\ \hline \mathbf{y}_2^T \end{array} \right) = \lambda_1 \mathbf{x}_1 \mathbf{y}_1^T + \lambda_2 \mathbf{x}_2 \mathbf{y}_2^T.$$

Thus in our case we have

$$\begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}^n = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \begin{pmatrix} 1/5 & 1/5 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \end{pmatrix}^n \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 2/5 & -3/5 \end{pmatrix}.$$

This expression emphasizes the fact that A^n converges to a rank 1 matrix:

$$A^n \to \begin{pmatrix} 3 \\ 2 \end{pmatrix} \begin{pmatrix} 1/5 & 1/5 \end{pmatrix} \text{ as } n \to \infty.$$

Remember: This section is just for motivation. I will explain all of the ideas later.

1.2 The Characteristic Polynomial

Let A be a square matrix over $\mathbb{R}$ or $\mathbb{C}$. We say that $\lambda \in \mathbb{C}$ is an *eigenvalue* of A when there exists a nonzero vector $\mathbf{x}$ satisfying $A\mathbf{x} = \lambda\mathbf{x}$. Let me emphasize this:

$$\boxed{\lambda \text{ is an eigenvalue of } A \quad \Longleftrightarrow \quad \text{there exists some } \mathbf{x} \neq \mathbf{0} \text{ satisfying } A\mathbf{x} = \lambda\mathbf{x}.}$$

If $A\mathbf{x} = \lambda\mathbf{x}$ then we say that $\mathbf{x}$ is a λ-*eigenvector* of A.

It is not immediately clear that eigenvalues exist. Our first result will show that any matrix has at least one eigenvalue. To do this we will rewrite the definition of eigenvalues in terms of determinants.[3] The following equivalences follow from results in the previous chapter:

$$\begin{aligned} \lambda \text{ is an eigenvalue of } A \quad & \Longleftrightarrow \quad A\mathbf{x} = \lambda\mathbf{x} \text{ for some } x \neq \mathbf{0} \\ & \Longleftrightarrow \quad (\lambda\mathbf{x} - A\mathbf{x}) = \mathbf{0} \text{ for some } \mathbf{x} \neq \mathbf{0} \\ & \Longleftrightarrow \quad \lambda I\mathbf{x} - A\mathbf{x} = \mathbf{0} \text{ for some } \mathbf{x} \neq \mathbf{0} \\ & \Longleftrightarrow \quad (\lambda I - A)\mathbf{x} = \mathbf{0} \text{ for some } \mathbf{x} \neq \mathbf{0} \\ & \Longleftrightarrow \quad \text{the nullspace } \mathcal{N}(\lambda I - A) \text{ contains a nonzero vector} \\ & \Longleftrightarrow \quad \dim \mathcal{N}(\lambda I - A) \geq 1 \\ & \Longleftrightarrow \quad \text{the matrix } \lambda I - A \text{ is not invertible} \\ & \Longleftrightarrow \quad \det(\lambda I - A) = 0. \end{aligned}$$

This last equivalence is the most convenient way to study eigenvalues. In summary,

$$\boxed{\lambda \text{ is an eigenvalue of } A \quad \Longleftrightarrow \quad \det(\lambda I - A) = 0.}$$

If λ is an eigenvalue of an $n \times n$ matrix A, we observe that the set of λ-eigenvectors is a subspace of $\mathbb{R}^n$. Indeed, the λ-eigenvectors of A are just the vectors in the nullspace $\mathcal{N}(\lambda I - A)$. We say that

$$\mathcal{N}(\lambda I - A) = \text{the } \lambda\text{-}\textit{eigenspace} \text{ of } A.$$

[3] There is a popular textbook called *Linear Algebra Done Right* in which the author goes to great lengths to avoid the use of determinants in the theory of eigenvalues. There is another well-known textbook called *Linear Algebra Done Wrong*, which I greatly prefer.

When λ is **not** an eigenvalue the matrix $\lambda I - A$ is invertible, so in this case the "λ-eigenspace" is trivial: $\mathcal{N}(\lambda I - A) = \{\mathbf{0}\}$. Before going further with the theory, we compute the eigenvalues of a general 2×2 matrix:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We have

$$\begin{aligned}
\det(\lambda I - A) &= \det\left(\lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \\
&= \det \begin{pmatrix} \lambda - a & -b \\ -c & \lambda - d \end{pmatrix} \\
&= (\lambda - a)(\lambda - d) - (-b)(-c) \\
&= \lambda^2 - (a+d)\lambda + (ad - bc).
\end{aligned}$$

Hence the eigenvalues of A are

$$\lambda = \frac{a + d \pm \sqrt{(a+d)^2 - 4(ad - bc)}}{2}.$$

Let $\Delta = (a+d)^2 - 4(ad - bc)$ denote the discriminant of this quadratic polynomial. If $\Delta = 0$ then the matrix A has only one eigenvalue. Now suppose that A has real entries. If $\Delta > 0$ then A has two distinct real eigenvalues and if $\Delta < 0$ then A has two distinct complex eigenvalues.

After finding the eigenvalues, it is an easy matter to find all of the eigenvectors. Consider our example from the previous section:

$$A = \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}.$$

The eigenvalues are the roots of the polynomial equation

$$\begin{aligned}
\det(\lambda I - A) &= 0 \\
\lambda^2 - (.8 + .7)\lambda + (.8)(.7) - (.2)(.3) &= 0 \\
\lambda^2 - 1.5\lambda + 0.5 &= 0,
\end{aligned}$$

which are

$$\begin{aligned}
\lambda &= \frac{1}{2}(1.5 \pm \sqrt{(1.5)^2 - 4(0.5)}) \\
&= \frac{1}{2}(1.5 \pm \sqrt{0.25}) \\
&= \frac{1}{2}(1.5 \pm 0.5) \\
&= 1 \text{ and } 1/2.
\end{aligned}$$

To find the 1-eigenvectors, we use row reduction to compute the nullspace of $1I - A$. First we observe that the matrix $1I - A$ has dependent rows (hence also dependent columns):

$$1I - A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix} = \begin{pmatrix} .2 & -.3 \\ -.2 & .3 \end{pmatrix}.$$

Indeed, this must be the case because 1 is an eigenvalue. Then we compute the RREF:

$$(1I - A)\mathbf{x} = \mathbf{0} \quad \rightsquigarrow \quad \begin{pmatrix} .2 & -.3 \\ -.2 & .3 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \overset{\text{RREF}}{\rightsquigarrow} \quad \begin{pmatrix} 1 & -3/2 \\ 0 & 0 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

It follows that there is a line of 1-eigenvectors:[4]

$$\mathbf{x} = t\begin{pmatrix} 3/2 \\ 1 \end{pmatrix}.$$

Next we compute the (1/2)-eigenspace:

$$\left(\frac{1}{2}I - A\right)\mathbf{x} = \mathbf{0} \quad \rightsquigarrow \quad \begin{pmatrix} -.3 & -.3 \\ -.2 & -.2 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \overset{\text{RREF}}{\rightsquigarrow} \quad \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus we have also have a line of (1/2)-eigenvectors:[5]

$$\mathbf{x} = t\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

The procedure is the same for larger matrices. Given the eigenvalues, we can find all of the eigenvectors by row reduction. The hard part is to find the eigenvalues.[6] In general, we define the *characteristic polynomial* of a square matrix A:

$$\boxed{\chi_A(\lambda) := \det(\lambda I - A).}$$

This is, indeed, a polynomial in λ. Furthermore, if A is $n \times n$ then $\chi_A(\lambda)$ is a polynomial of degree n. In general the coefficients are quite complicated, but two of the coefficients have special names. We have

$$\boxed{\chi_A(\lambda) = \lambda^n - \operatorname{tr}(A)\lambda^{n-1} + \cdots + (-1)^n \det(A),}$$

where the *trace* of a square matrix is defined as the sum of its diagonal entries:

$$\boxed{\operatorname{tr}(A) = \operatorname{tr}\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} := a_{11} + a_{22} + \cdots + a_{nn}.}$$

[4] In the previous section I chose $\mathbf{x} = (3, 2)$ to avoid fractions.

[5] In the previous example I chose $\mathbf{x} = (1, -1)$ because I didn't want a negative sign in the first coordinate.

[6] Solving polynomials equations is a non-linear problem. There are no exact algorithms, but there are reasonably good approximation schemes. The state of the art for computing eigenvalues is the *QR algorithm*, which doesn't use the characteristic polynomial at all.

We already know this formula for 2×2 matrices, and the general case is not hard to check.

If matrices A and B satisfy $B = XAX^{-1}$ for some invertible matrix X, then I claim that A and B have the same characteristic polynomial:

$$\boxed{\chi_{XAX^{-1}}(\lambda) = \chi_A(\lambda).}$$

To prove this, we note that

$$\begin{aligned}
\chi_B(\lambda) &= \det(\lambda I - B) \\
&= \det(\lambda XX^{-1} - XAX^{-1}) \\
&= \det(X(\lambda I - A)X^{-1}) \\
&= \cancel{\det(X)}\det(\lambda I - A)\cancel{\det(X)^{-1}} \\
&= \det(\lambda I - A) \\
&= \chi_A(\lambda).
\end{aligned}$$

By comparing the coefficients of $\chi_A(\lambda)$ and $\chi_B(\lambda)$, it follows that[7]

$$\boxed{\mathrm{tr}(XAX^{-1}) = \mathrm{tr}(A) \quad \text{and} \quad \det(XAX^{-1}) = \det(A).}$$

The eigenvalues of a square matrix A are the roots of the characteristic polynomial. It follows from the Fundamental Theorem of Algebra that

$$\boxed{\text{Every square matrix has at least one eigenvalue.}}$$

Indeed, since the characteristic polynomial $\chi_A(\lambda)$ has degree n, the FTA says that there exist complex numbers $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ such that

$$\chi_A(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n).$$

We can expand this to get

$$\boxed{\chi_A(\lambda) = \lambda^n - (\lambda_1 + \cdots + \lambda_n)\lambda^{n-1} + \cdots + (-1)^n \lambda_1 \cdots \lambda_n.}$$

Then comparing the coefficients with our previous expansion for $\chi_A(\lambda)$ gives

$$\boxed{\mathrm{tr}(A) = \lambda_1 + \cdots + \lambda_n \quad \text{and} \quad \det(A) = \lambda_1 \cdots \lambda_n.}$$

That is, the trace of A equals the sum of the eigenvalues (with multiplicities) and the determinant of A equals the product of the eigenvalues (with multiplicities). This is often useful.

Remarks:

[7] Of course, we already knew this property of the determinant.

- I guess we could say that every $n \times n$ matrix has n eigenvalues, but they need not be distinct. For example, the identity matrix I_n has characteristic polynomial

$$\det(\lambda I_n - I_n) = \det \begin{pmatrix} \lambda - 1 & & \\ & \ddots & \\ & & \lambda - 1 \end{pmatrix} = (\lambda - 1)^n,$$

hence 1 is the only eigenvalue. The corresponding eigenspace is all of $\mathbb{R}^n$. Indeed, every vector $\mathbf{x} \in \mathbb{R}^n$ is a 1-eigenvector of the identity matrix: $I_n\mathbf{x} = \mathbf{x} = 1\mathbf{x}$.

- A real matrix has at least one complex eigenvalue, but it need not have any real eigenvalues. For example, consider the matrix that rotates counterclockwise by 90°:

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The characteristic polynomial is

$$\det(\lambda I - A) = \begin{pmatrix} \lambda & 1 \\ -1 & \lambda \end{pmatrix} = \lambda^2 + 1,$$

hence the eigenvalues are $\pm i$. The corresponding eigenspaces are

$$\mathcal{N}(iI - A) = t\begin{pmatrix} 1 \\ -i \end{pmatrix} \quad \text{and} \quad \mathcal{N}(-iI - A) = t\begin{pmatrix} 1 \\ i \end{pmatrix}.$$

Hence A is a real matrix with no real eigenvalues and no real eigenvectors.

1.3 Diagonalization

We say that a square matrix A is *diagonalizable* when it has a basis of eigenvectors. So far we have seen only diagonalizable matrices. Here is the simplest example of a matrix that is **not diagonalizable**:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The characteristic polynomial is

$$\det(\lambda I - A) = \det \begin{pmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 1 \end{pmatrix} = (\lambda - 1)^2,$$

hence 1 is the only eigenvalue. But the 1-eigenspace is only one dimensional:

$$(1I - A)\mathbf{x} = \mathbf{0} \quad \rightsquigarrow \quad \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightsquigarrow \quad \mathbf{x} = t\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Non-diagonalizable matrices are quite a nuisance. Fortunately, they are rare. The next result shows that any $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Theorem (Distinct Eigenvalues Implies Diagonalizable). Let A be an $n \times n$ matrix. Suppose that the characteristic polynomial factors as

$$\chi_A(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n),$$

where the complex numbers $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ are distinct. Furthermore, let $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{C}^n$ be some nonzero vectors satisfying $A\mathbf{x}_i = \lambda_i \mathbf{x}_i$.[8] Then I claim that $\mathbf{x}_1, \ldots, \mathbf{x}_n$ is a basis for $\mathbb{C}^n$.

Warning: This theorem is not sharp. If the characteristic polynomial of a matrix has a repeated factor then the matrix may or may not be diagonalizable. For example, the following two matrices both have characteristic polynomial $(\lambda - 1)^2$:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The matrix on the left is not diagonalizable, but the matrix on the right **is** diagonalizable. Indeed, every vector is a 1-eigenvector for the identity matrix. I will give a sharp characterization of diagonalizability in the next section.

Proof. It is enough to show that the set $\mathbf{x}_1, \ldots, \mathbf{x}_n$ is linearly independent. Then the subspace of $\mathbb{C}^n$ spanned by $\mathbf{x}_1, \ldots, \mathbf{x}_n$ is n-dimensional, hence it must be the whole space.

First we observe that the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are distinct. Indeed, suppose we had $\mathbf{x}_i = \mathbf{x}_j = \mathbf{x}$ for some $i \neq j$. This would imply that

$$\begin{aligned} A\mathbf{x} &= A\mathbf{x} \\ \lambda_i \mathbf{x} &= \lambda_j \mathbf{x} \\ (\lambda_i - \lambda_j)\mathbf{x} &= \mathbf{0}. \end{aligned}$$

But by assumption we have $\lambda_i - \lambda_j \neq 0$ and $\mathbf{x} \neq \mathbf{0}$, which gives a contradiction. This is no big deal; it just says that a given vector can't be an eigenvector for two different eigenvalues.

We will prove by induction on k that the set of vectors $\mathbf{x}_1, \ldots, \mathbf{x}_k$ is independent for any $1 \leq k \leq n$, and it will follow that the set $\mathbf{x}_1, \ldots, \mathbf{x}_n$ is independent. The result is trivial for $k = 1$ because any set containing one vector is by convention called independent. It is not logically necessary, but let's also consider the case $k = 2$, to get a feel for the general argument. Suppose that we have

$$b_1\mathbf{x}_1 + b_2\mathbf{x}_2 = \mathbf{0}$$

for some scalars $b_1, b_2 \in \mathbb{C}$. Our goal is to show that $b_1 = 0$ and $b_2 = 0$. First multiply both sides on the left by A to obtain

$$\begin{aligned} A(b_1\mathbf{x}_1 + b_2\mathbf{x}_2) &= A\mathbf{0} \\ b_1 A\mathbf{x}_1 + b_2 A\mathbf{x}_2 &= \mathbf{0} \end{aligned}$$

[8] Such vectors exist because $\lambda_1, \ldots, \lambda_n$ are eigenvalues.

$$b_1\lambda_1\mathbf{x}_1 + b_2\lambda_2\mathbf{x}_2 = \mathbf{0}.$$

Subtract λ_2 times the first equation from this equation to obtain

$$\begin{aligned}
(b_1\lambda_1\mathbf{x}_1 + b_2\lambda_2\mathbf{x}_2) - \lambda_2(b_1\mathbf{x}_1 + b_2\mathbf{x}_2) &= \mathbf{0} \\
b_1(\lambda_1 - \lambda_2)\mathbf{x}_1 + b_2(\lambda_2 - \lambda_2)\mathbf{x}_2 &= \mathbf{0} \\
b_1(\lambda_1 - \lambda_2)\mathbf{x}_1 &= \mathbf{0}.
\end{aligned}$$

Since $\lambda_1 - \lambda_2 \neq 0$ and $\mathbf{x}_1 \neq \mathbf{0}$ this implies that $b_1 = 0$. But then substituting into the first equation gives

$$\begin{aligned}
b_1\mathbf{x}_1 + b_2\mathbf{x}_2 &= \mathbf{0} \\
0\mathbf{x}_1 + b_2\mathbf{x}_2 &= \mathbf{0} \\
b_2\mathbf{x}_2 &= \mathbf{0},
\end{aligned}$$

which implies that $b_2 = 0$ because $\mathbf{x}_2 \neq \mathbf{0}$.

Now we prove the general case. Fix some $k \geq 2$ and suppose that we have

$$b_1\mathbf{x}_1 + b_2\mathbf{x}_2 + \cdots + b_k\mathbf{x}_k = \mathbf{0} \tag{1}$$

for some scalars $b_1, \ldots, b_k \in \mathbb{C}$. Our goal is to show that $b_1 = b_2 = \cdots = b_k = 0$. To do this we multiply both sides on the left by A to obtain

$$\begin{aligned}
A(b_1\mathbf{x}_1 + b_2\mathbf{x}_2 + \cdots + b_k\mathbf{x}_k) &= A\mathbf{0} \\
b_1A\mathbf{x}_1 + b_2A\mathbf{x}_2 + \cdots + b_kA\mathbf{x}_k &= \mathbf{0} \\
b_1\lambda_1\mathbf{x}_1 + b_2\lambda_2\mathbf{x}_2 + \cdots + b_k\lambda_k\mathbf{x}_k &= \mathbf{0}. \qquad (2)
\end{aligned}$$

Then we consider the equation $(2) - \lambda_k(1)$:

$$\begin{aligned}
\left(\sum_{i=1}^{k} b_i\lambda_i\mathbf{x}_i\right) - \lambda_k\left(\sum_{i=1}^{k} b_i\mathbf{x}_i\right) &= \mathbf{0} \\
\sum_{i=1}^{k} b_i(\lambda_i - \lambda_k)\mathbf{x}_k &= \mathbf{0} \\
0\mathbf{x}_k + \sum_{i=1}^{k-1} b_i(\lambda_i - \lambda_k)\mathbf{x}_i &= \mathbf{0} \\
\sum_{i=1}^{k-1} b_i(\lambda_i - \lambda_k)\mathbf{x}_i &= \mathbf{0}.
\end{aligned}$$

By induction, the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}$ are independent, hence for any $1 \leq i \leq k-1$ we must have $b_i(\lambda_i - \lambda_k) = 0$. But by assumption we have $\lambda_i \neq \lambda_k$, and hence $b_i = 0$, for any $1 \leq i \leq k-1$. Finally, we substitute back into equation (1) to obtain $0\mathbf{x}_1 + \cdots + 0\mathbf{x}_{k-1} + b_k\mathbf{x}_k = \mathbf{0}$, which implies that $b_k = 0$ because $\mathbf{x}_k \neq \mathbf{0}$. $\square$

This result implies that "almost all" matrices are diagonalizable. To see this, we will use the concept of the *discriminant* of a polynomial. For example, consider the general 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The characteristic polynomial is

$$\chi_A(\lambda) = \lambda^2 - (a+d)\lambda + (ad - bc),$$

and hence the eigenvalues are

$$\lambda = \frac{a + d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}.$$

The quantity $\Delta(a,b,c,d) = (a+d)^2 - 4(ad-bc)$ is called the discriminant of the characteristic polynomial. If $\Delta \neq 0$ then we observe that A has two distinct eigenvalues, hence is diagonalizable. If we choose the entries a, b, c, d of the matrix A at random then it would be quite unlikely to have $\Delta(a,b,c,d) = 0$. To be more precise, we can view the set of 2×2 matrices as a 4-dimensional vector space:

$$\mathbb{C}^{2\times 2} = \text{the vector space of } 2 \times 2 \text{ matrices with complex entries.}$$

Inside this 4-dimensional vector space, the set of matrices satisfying $\Delta(a,b,c,d) = 0$ forms a "3-dimensional subset".[9] By analogy, consider a 2-dimensional plane in $\mathbb{R}^3$. A randomly chosen point in $\mathbb{R}^3$ will not lie on this plane. Similarly, a randomly chosen point in a 4-dimensional vector space will not lie in a given 3-dimensional shape.

This discussion generalizes to square matrices of any size. Given an $n \times n$ matrix A with entries a_{ij}, there is a certain polynomial $\Delta(A)$ in the n^2 variables a_{ij} such that $\Delta(A) = 0$ if and only if A has a repeated eigenvalue. Since the equation $\Delta(A) = 0$ defines an $(n^2 - 1)$-dimensional subset of the n^2-dimensional space of $n \times n$ matrices, a randomly chosen matrix will have distinct eigenvalues, and hence will be diagonalizable.

Why do we call it diagonalization? Let A be a diagonalizable $n \times n$ matrix and let $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{C}^n$ be a basis of eigenvectors with corresponding eigenvalues $A\mathbf{x}_i = \lambda_i \mathbf{x}_i$. (Here we do not assume that the eigenvalues are distinct.) We can write the n equations $A\mathbf{x}_i = \lambda_i \mathbf{x}_i$ simultaneously as a matrix equation:

$$\begin{aligned} \left(\begin{array}{c|c|c} A\mathbf{x}_1 & \cdots & A\mathbf{x}_n \end{array} \right) &= \left(\begin{array}{c|c|c} \lambda_1\mathbf{x}_1 & \cdots & \lambda_n\mathbf{x}_n \end{array} \right) \\ A \left(\begin{array}{c|c|c} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{array} \right) &= \left(\begin{array}{c|c|c} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{array} \right) \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \end{aligned}$$

[9] For a general polynomial $f(x_1, \ldots, x_n)$ in n variables, the set of points $\mathbf{x} \in \mathbb{C}^n$ satisfying $f(\mathbf{x}) = 0$ forms an $(n-1)$-dimensional subset. I don't want to be too precise about this.

$$AX = X\Lambda,$$

where Λ is a **diagonal** matrix containing the eigenvalues. Since the columns of X are independent by assumption, the matrix X is invertible and we can write

$$\boxed{A = X\Lambda X^{-1}} \quad \text{or} \quad \boxed{X^{-1}AX = \Lambda.}$$

Thus we have used the eigenvector matrix X to convert A into the diagonal matrix Λ. In other words, we have "diagonalized" A. We will see below that this is extremely useful.

1.4 Evaluating a Polynomial at a Matrix

Matrices can be added, multiplied by scalars, and raised to powers. This allows us to consider polynomials of matrices. More precisely, we can "evaluate" polynomials at matrices. Consider a polynomial in one variable, with complex coefficients:

$$f(x) = b_0 + b_1 x + \cdots + b_k x^k.$$

Then for any $n \times n$ matrix A we define the $n \times n$ matrix $f(A)$ by[10]

$$f(A) := b_0 I_n + b_1 A + b_2 A^2 + \cdots + b_k A^k.$$

This evaluation behaves well with respect to eigenvalues and eigenvectors. That is, for any vector $\mathbf{x}$ and scalar λ, we have

$$\boxed{A\mathbf{x} = \lambda\mathbf{x} \quad \Longrightarrow \quad f(A)\mathbf{x} = f(\lambda)\mathbf{x}.}$$

Indeed, if $A\mathbf{x} = \lambda\mathbf{x}$ then we can show by induction that $A^m\mathbf{x} = \lambda^m\mathbf{x}$ for any $m \geq 0$:

- **Base Case.** $A^0\mathbf{x} = I_n\mathbf{x} = \mathbf{x} = \lambda^0\mathbf{x}$.
- **Induction Step.** If $A^{m-1}\mathbf{x} = \lambda^{m-1}\mathbf{x}$ then

$$A^m\mathbf{x} = A(A^{m-1}\mathbf{x}) = A^m(\lambda^{m-1}\mathbf{x}) = \lambda^{m-1}(A\mathbf{x}) = \lambda^{m-1}\lambda\mathbf{x} = \lambda^m\mathbf{x}.$$

Then for any polynomial $f(x) = b_0 + b_1 x + \cdots + b_k x^k$ we have

$$\begin{aligned} f(A)\mathbf{x} &= (b_0 I_n + b_1 A + \cdots + b_k A^k)\mathbf{x} \\ &= b_0 I_n\mathbf{x} + b_1 A\mathbf{x} + \cdots + b_k A^k\mathbf{x} \\ &= b_0\mathbf{x} + b_1\lambda\mathbf{x} + \cdots + b_k\lambda^k\mathbf{x} \\ &= (b_0 + b_1\lambda + \cdots + b_k\lambda^k)\mathbf{x} \\ &= f(\lambda)\mathbf{x}. \end{aligned}$$

[10]Here we use the convention that $A^0 = I_n$.

Now suppose that the matrix A is a "root" of the polynomial $f(x)$. That is, suppose that $f(A)$ is the zero matrix. Then every eigenvalue of A is also a root of $f(x)$:

$$\boxed{f(A) = O \quad \Longrightarrow \quad \text{every eigenvalue of } A \text{ satisfies } f(\lambda) = 0.}$$

Indeed, if $A\mathbf{x} = \lambda\mathbf{x}$ and $f(A) = O$ then we have

$$f(\lambda)\mathbf{x} = f(A)\mathbf{x} = O\mathbf{x} = \mathbf{0}.$$

And if $\mathbf{x} \neq \mathbf{0}$ then this implies $f(\lambda) = 0$. Here are some examples.

Projections. Any matrix satisfying $P^2 = P$ has eigenvalues in the set $\{0, 1\}$. Indeed, if $P^2 - P = O$ then any eigenvalue λ of P satisfies

$$\begin{aligned} \lambda^2 - \lambda &= 0 \\ \lambda(\lambda - 1) &= 0 \\ \lambda &= 0 \text{ or } 1. \end{aligned}$$

This doesn't mean that both eigenvalues must occur. For example, the zero matrix satisfies $O^2 = O$ and its only eigenvalue is 0, while the identity matrix satisfies $I^2 = I$ and its only eigenvalue is 1.

At the end of this section we will show that any matrix satisfying $P^2 = P$ is diagonalizable. Assuming this for now, we can prove that any $n \times n$ matrix satisfying $P^2 = P$ is a (possibly non-orthogonal) projection matrix. To do this, let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be a basis of eigenvectors. Since the only possible eigenvalues are 1 and 0, we can sort the eigenvectors so that $P\mathbf{x}_i = 1\mathbf{x}_i = \mathbf{x}_i$ for $0 \leq i \leq r$ and $P\mathbf{x}_i = 0\mathbf{x}_i = \mathbf{0}$ for $r < i \leq n$.[11] This gives the factorization

$$P = X \left(\begin{array}{c|c} I_r & O_{r,n-r} \\ \hline O_{n-r,r} & O_{n-r}, n-r \end{array} \right) X^{-1}.$$

Now let A be the $n \times r$ matrix consisting of the first r columns of X and let B be the $r \times n$ matrix consisting of the first r rows of X^{-1}. Then we have

$$P = \left(\begin{array}{c|c} A & * \end{array} \right) \left(\begin{array}{c|c} I_r & O_{r,n-r} \\ \hline O_{n-r,r} & O_{n-r}, n-r \end{array} \right) \left(\begin{array}{c} B \\ \hline * \end{array} \right) = \left(\begin{array}{c|c} A & * \end{array} \right) \left(\begin{array}{c} B \\ \hline O \end{array} \right) = AB + O = AB.$$

And we also have

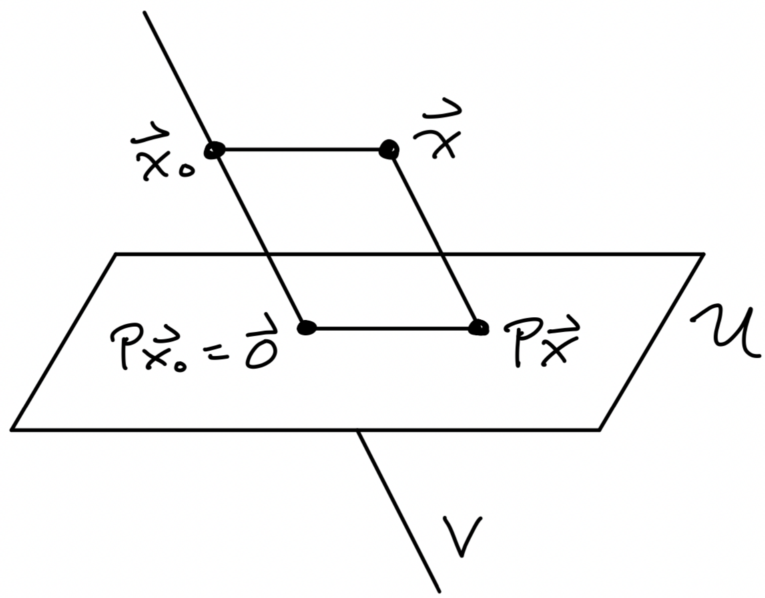
$$\left(\begin{array}{c|c} I_r & O_{r,n-r} \\ \hline O_{n-r,r} & I_{n-r,n-r} \end{array} \right) = I_n = X^{-1}X = \left(\begin{array}{c} B \\ \hline * \end{array} \right) \left(\begin{array}{c|c} A & * \end{array} \right) = \left(\begin{array}{c|c} BA & * \\ \hline * & * \end{array} \right),$$

which implies that $BA = I_r$. In summary:

$$\boxed{P^2 = P \quad \Longrightarrow \quad P = AB \text{ for some } A, B \text{ satisfying } BA = I_r, \text{ where } r = \text{rank}(P).}$$

This is the projection onto the column space $U = \mathcal{C}(A)$, in a direction parallel to the null space $V = \mathcal{N}(B)$. Picture:

[11] We allow the possibilities $r = 0$ (all eigenvalues are 0) and $r = n$ (all eigenvalues are 1).



The projection is orthogonal if and only if $V = U^\perp$. In this case we have $\mathcal{N}(B) = \mathcal{C}(A)^\perp = \mathcal{N}(A^T)$, which impliess that $\mathcal{R}(A) = \mathcal{N}(B)^T = \mathcal{N}(A^T)^\perp = \mathcal{R}(A^T)$. Since $\mathcal{R}(B) = \mathcal{R}(A^T)$ we can find an invertible $r \times r$ matrix S of row operations such that $B = SA^T$. But then $BA = I_r$ implies $SA^TA = I_r$ and hence $S = (A^TA)^{-1}$. Finally, we conclude that

$$P = AB = ASA^T = A(A^TA)^{-1}A^T,$$

which agrees with our previous formula for orthogonal projections.

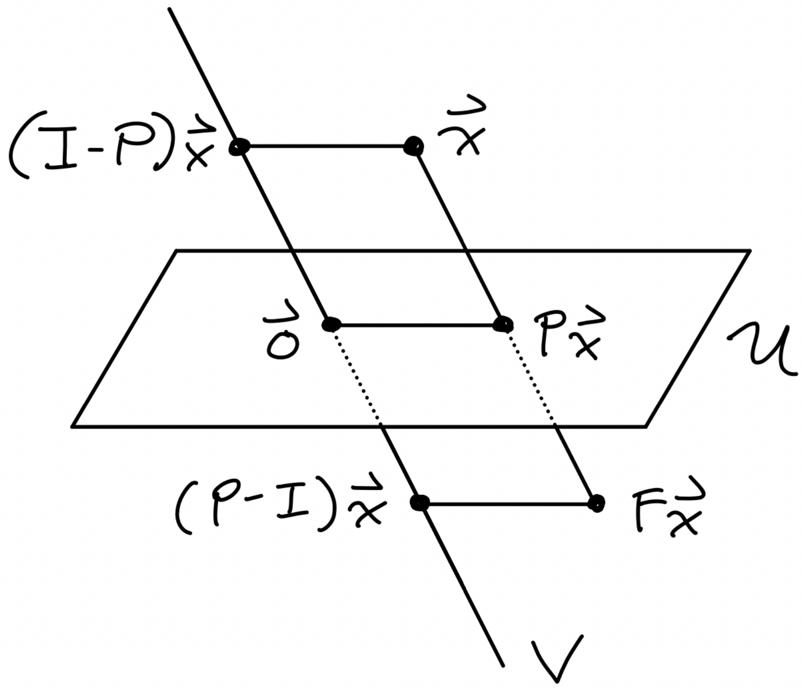
Reflections. Any matrix satisfying $F^2 = I$ has eigenvalues in the set $\{1, -1\}$. Indeed, if $F^2 - I = O$ then any eigenvalue λ of F satisfies

$$\begin{aligned} \lambda^2 - 1 &= 0 \\ \lambda^2 &= 1 \\ \lambda &= 1 \text{ or } -1. \end{aligned}$$

Consider the unique matrix P satisfying $F = 2P - I$ and $P = (F + I)/2$. We observe that

$$P^2 = \frac{1}{4}(F^2 + 2F + I^2) = \frac{1}{2}(I + 2F + I) = \frac{1}{4}(2F + 2I) = \frac{1}{2}(F + I) = P,$$

so that P is a projection. Let U and V be the 1-eigenspace and 0-eigenspace of P as in the previous example, then U is the 1-eigenspace of F and V is the (-1)-eigenspace of F. Geometrically, F is the reflection across the subspace U in the direction of V. Picture:



In terms of matrices, if F is a matrix of rank r satisfying $F^2 = I$ then we can find two $r \times (n-r)$ matrices A, B satisfying $BA = I_r$, such that

$$F = 2P - I = 2AB - I.$$

This is the reflection across the column space $U = \mathcal{C}(A)$, parallel to the nullspace $V = \mathcal{N}(B)$.

Rotations. Any matrix satisfying $R^n = I$ has eigenvalues in the set $\{e^{2\pi i k/n} : k \in \mathbb{Z}\}$. Indeed, since $R^n - I = O$, any eigenvalue λ of R must satisfy $\lambda^n - 1 = 0$, and hence must be an nth root of unity. Such matrices can be quite complicated. For a simple example, we consider the 2×2 rotation matrix:

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

The characteristic polynomial is

$$\chi_{R_\theta}(\lambda) = \lambda^2 - 2\cos\theta\lambda + 1,$$

hence the eigenvalues are[12]

$$\begin{aligned} \lambda &= \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} \\ &= \frac{2\cos\theta \pm 2\sqrt{\cos^2\theta - 1}}{2} \end{aligned}$$

[12] It's a bit reckless to take square roots in this way, but it gives the correct answer.

$$
\begin{aligned}
&= \frac{2\cos\theta \pm 2\sqrt{-\sin^2\theta}}{2} \\
&= \frac{2\cos\theta \pm 2i\sin\theta}{2} \\
&= \cos\theta \pm i\sin\theta \\
&= e^{\pm i\theta}.
\end{aligned}
$$

The case $\theta = 0$ corresponds to the identity matrix, with eigenvalues $(1, 1)$ and the case $\theta = \pi$ corresponds to the negative identity matrix, with eigenvalues $(-1, -1)$. In all other cases, the eigenvalues (and hence also the eigenvectors) are not real. If $\theta = 2\pi/n$ then the eigenvalues $\lambda = e^{\pm 2\pi i/n}$ satisfy $\lambda^n = 1$. This agrees with the fact that

$$(R_{2\pi/n})^n = I.$$

Next, we give an alternative proof for the existence of eigenvalues, which does not use determinants.[13]

Theorem (Existence of Eigenvalues). Let A be any $n \times n$ matrix with real or complex entries and consider an arbitrary nonzero vector $\mathbf{v} \in \mathbb{C}^n$. Since $\mathbb{C}^n$ is n-dimensional, the following $n + 1$ vectors must be linearly dependent:[14]

$$\mathbf{v}, A\mathbf{v}, A^2\mathbf{x}, \ldots, A^n\mathbf{v}.$$

In other words, we can find scalars $b_0, b_1, \ldots, b_n$, not all zero, such that

$$b_0\mathbf{v} + b_1A\mathbf{v} + b_2A^2\mathbf{v} + \cdots + b_nA^n\mathbf{v} = \mathbf{0}.$$

In fact, one of the scalars $b_1, \ldots, b_n$ must be nonzero, otherwise we would have $b_0\mathbf{v} = \mathbf{0}$ and $b_0 \neq 0$, which contradicts the fact that $\mathbf{v} \neq \mathbf{0}$. We can rewrite the previous equation as

$$
\begin{aligned}
(b_0I + b_1A + b_2A^2 + \cdots + b_nA^n)\mathbf{v} &= \mathbf{0}, \\
f(A) &= \mathbf{0},
\end{aligned}
$$

for the polynomial $f(x) = b_0 + b_1x + b_2x^2 + \cdots b_nx^n$, which has degree between 1 and n because not all of $b_1, \ldots, b_n$ are zero. Let's say $\deg(f) = k$. By the Fundamental Theorem of Algebra we can factor $f(x)$ as

$$f(x) = (x - \alpha_1)(x - \alpha_2)\cdots(x - \alpha_k),$$

for some complex numbers $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$, not necessarily distinct. Now the equation $f(A)\mathbf{v} = \mathbf{0}$ becomes[15]

$$(A - \alpha_1I)(A - \alpha_2I)\cdots(A - \alpha_kI)\mathbf{v} = \mathbf{0}.$$

[13]This proof is the motivation for Axler's approach in *Linear Algebra Done Right*.

[14]Indeed, **any** collection of $n + 1$ vectors in $\mathbb{C}^n$ is linearly dependent.

[15]There is a subtle point hiding here. Given a polynomial $f(x)$ and square matrices A, B, it is **not** generally true that $f(AB) = f(A)f(B)$. However, if $AB = BA$ then we do have $f(AB) = f(A)f(B)$. Since the matrices $A - \alpha_1I, \ldots, A - \alpha_kI$ commute with each other, we are okay in this case.

To save notation, let's write $A_i = A - \alpha_i I$. Thus we have

$$A_1 A_2 \cdots A_k \mathbf{v} = \mathbf{0}.$$

Since $\mathbf{v} \neq \mathbf{0}$, this implies that the matrix $A_1 \cdots A_k$ is not invertible. And since a product of invertible matrices is invertible, this implies that at least one of the factors, say A_i, is not invertible. Finally, since $A_i = A - \alpha_i I$ is not invertible, we conclude that α_i is an eigenvalue of A. In particular, we have shown that A has an eigenvalue. $\square$

Building on this idea, we can give a sharper result about diagonalization. The proof is tricky so you can feel free to skip it.

Theorem (Existence of Diagonalization). A square matrix A is diagonalizable if and only if we have $f(A) = O$ for some polynomial with no repeated roots.

Proof. First suppose that A has a basis of eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ with corresponding eigenvalues $A\mathbf{x}_i = \lambda_i \mathbf{x}_i$. Some of these eigenvalues might be repeated. Let $\mu_1, \ldots, \mu_k$ be the list of eigenvalues with repetition removed, and consider the polynomial

$$f(x) = (x - \mu_1) \cdots (x - \mu_k),$$

which has no repeated roots. We will show that $f(A) = O$. To do this, we observe that the matrices $(A - \mu_i I)$ and $(A - \mu_j I)$ commute for any i, j:

$$(A - \mu_i I)(A - \mu_j I) = A^2 - (\mu_i + \mu_j)A + \mu_i \mu_j I = (A - \mu_j I)(A - \mu_i - I).$$

We will use this to show that $f(A)\mathbf{v} = \mathbf{0}$ for any eigenvector $\mathbf{v}$. Then since there exists a basis of eigenvectors, it will follow from this that $f(A)\mathbf{v} = \mathbf{0}$ for **any** vector $\mathbf{v}$, and hence $f(A)$ is the zero matrix. So let $\mathbf{v}$ be an eigenvector with eigenvalue μ_i.[16] Then we have[17]

$$\begin{aligned}
f(A)\mathbf{v} &= \left(\prod_j (A - \mu_j I) \right) \mathbf{v} \\
&= \left(\prod_{j \neq i} (A - \mu_j) \right) (A - \mu_i)\mathbf{v} \\
&= \left(\prod_{j \neq i} (A - \mu_j) \right) (A\mathbf{v} - \mu_{\mathbf{v}}) \\
&= \left(\prod_{j \neq i} (A - \mu_j) \right) \mathbf{0} \\
&= \mathbf{0}.
\end{aligned}$$

[16]Recall that every eigenvalue is in the list $\mu_1, \ldots, \mu_k$.

[17]We used commutativity to pull the factor $A - \mu_i I$ to the right.

Thus we have shown that a diagonalizable matrix A satisfies an equation $f(A) = O$ for some polynomial $f(x)$ with no repeated roots.

Conversely, suppose that an $n \times n$ matrix A satisfies $f(A) = O$ for some polynomial $f(x)$ with no repeated roots. Suppose that $\deg(f) = k$ and write

$$f(x) = (x - \lambda_1) \cdots (x - \lambda_k)$$

for some distinct complex numbers $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$. We want to show that A has a basis of eigenvectors. First we define the null spaces

$$E_{\lambda_i} = \mathcal{N}(A - \lambda_i I) = \{\mathbf{x} : A\mathbf{x} = \lambda_i \mathbf{x}\}.$$

Note that $E_{\lambda_i} \neq \{\mathbf{0}\}$ if and only if λ_i is an eigenvalue, in which case E_{λ_i} is the corresponding eigenspace. We don't really care if all of the numbers λ_i are eigenvalues. Indeed, some of them might not be. Our goal is merely to show that the spaces $E_{\lambda_1}, \ldots, E_{\lambda_k}$ are big enough to fill up all of $\mathbb{C}^n$. To be precise, we will show that

- $E_{\lambda_i} \cap E_{\lambda_j} = \{\mathbf{0}\}$ for all $i \neq j$,
- $\mathbb{C}^n = \{\mathbf{x}_1 + \cdots + \mathbf{x}_k : \mathbf{x}_i \in E_{\lambda_i} \text{ for all } i\}$.

Then by concatenating bases for $E_{\lambda_1}, \ldots, E_{\lambda_k}$ we will obtain a basis for $\mathbb{C}^n$ that consists of eigenvectors of A. For the first statement, suppose that $\mathbf{x} \in E_{\lambda_i} \cap E_{\lambda_j}$ so that $\lambda_i \mathbf{x} = A\mathbf{x} = \lambda_j \mathbf{x}$. If $\mathbf{x} \neq \mathbf{0}$ then this implies that $\lambda_i = \lambda_j$ and hence $i \neq j$, because the λ_i are distinct. The second statement is trickier. First we consider the partial fraction expansion of $1/f(x)$:

$$\frac{1}{f(x)} = \frac{1}{(x - \lambda_1) \cdots (x - \lambda_k)} = \sum_i \frac{\alpha_i}{x - \lambda_i},$$

for some scalars $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$, not necessarily distinct.[18] Now consider the polynomials

$$p_i(x) = \frac{\alpha_i f(x)}{x - \lambda_i} = \alpha_i \prod_{j \neq i} (x - \lambda_j),$$

and note that

$$\begin{aligned}
p_1(x) + \cdots p_k(x) &= \frac{\alpha_1 f(x)}{x - \lambda_1} + \cdots + \frac{\alpha_k f(x)}{x - \lambda_k} \\
&= f(x) \cdot \sum_i \frac{\alpha_i}{x - \lambda_i} \\
&= f(x) \cdot \frac{1}{f(x)} \\
&= 1.
\end{aligned}$$

[18] I won't prove the existence of the partial fraction expansion. It depends on the theory of greatest common divisors in the ring of polynomials.

Finally, consider any vector $\mathbf{x} \in \mathbb{C}^n$ and write $\mathbf{x}_i := p_i(A)\mathbf{x}$. On the one hand, by evaluating the polynomial equation $(x - \lambda_i)p_i(A) = \alpha_i f(x)$ at A we have

$$(A - \lambda_i I)\mathbf{x}_i = (A - \lambda_i I)p_i(A)\mathbf{x} = \alpha_i f(A)\mathbf{x} = O\mathbf{x} = \mathbf{0},$$

and hence $\mathbf{x}_i \in E_{\lambda_i}$. On the other hand, by evaluating the polynomial equation $p_1(x) + \cdots + p_k(x) = 1$ at A we have

$$\begin{aligned}\mathbf{x}_1 + \cdots + \mathbf{x}_k &= p_1(A)\mathbf{x} + \cdots + p_k(A)\mathbf{x} \\ &= (p_1(A) + \cdots p_k(A))\mathbf{x} \\ &= I\mathbf{x} \\ &= \mathbf{x},\end{aligned}$$

as desired. $\square$

That was a tricky proof, but it's a useful theorem. In particular, it implies that any matrix satisfying $P^2 = P$, and hence $P^2 - P = O$, is diagonalizable because the polynomial $x^2 - x = x(x-1)$ has distinct roots. Furthermore, any matrix satisfying $R^n = I$, and hence $R^n - I = O$, is diagonalizable because the polynomial $x^n - 1$ has distinct roots:

$$x^n - 1 = (x-1)(x - e^{2\pi i/n})(x - e^{4\pi i/n}) \cdots (x - e^{2\pi i(n-1)/n}).$$

Finally, we examine what goes wrong for a specific non-diagonalizable matrix. Consider the following small matrices with repeated eigenvalues:

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

Each of these has characteristic polynomial $(x - \lambda)^2$:

$$(x-\lambda)^2 = \det \begin{pmatrix} x-\lambda & 0 \\ 0 & x-\lambda \end{pmatrix} = \det \begin{pmatrix} x-\lambda & -1 \\ 0 & x-\lambda \end{pmatrix}.$$

In the next section we will show that every matrix satisfies its own characteristic polynomial, which we can easily verify for these two matrices:

$$(A - \lambda I)^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad (B - \lambda I)^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The matrix A also satisfies the polynomial $f(x) = x - \lambda$, which has no repeated roots. This confirms that A is diagonalizable; in fact, it is diagonal. On the other hand, the matrix B is not diagonalizable. This is easy to check directly. Instead, we will prove it indirectly, by showing that B cannot satisfy any polynomial with distinct roots. The basic reason is that

$$(B - \lambda I)^2 = O \quad \text{but} \quad B - \lambda I \neq O.$$

Indeed, consider any polynomial $g(x) = (x - \lambda_1) \cdots (x - \lambda_k)$ with distinct roots $\lambda_1, \ldots, \lambda_k$. If $g(B) = O$ then one of the matrices $B - \lambda_j I$ must be non-invertible, so that λ_j is an eigenvalue and hence $\lambda_j = \lambda$. Since the λ_i are distinct, this implies that the λ_i with $i \neq j$ are **not** eigenvalues. Then since $g(B) = O$ equals $B - \lambda I$ times a product of invertible matrices $B - \lambda_i I$ for $i \neq j$, we conclude that $B - \lambda I = O$. Contradiction.

Remark: In the next chapter we will say more about non-diagonalizable matrices.

1.5 The Functional Calculus

Why are diagonalizable matrices good? As we mentioned in the first section, if we can diagonalize a matrix A then we can find explicit formulas for the entries of its powers A^k. More generally, diagonalizing a matrix allows us to compute any polynomial evaluation of the matrix $f(A)$. We can even compute convergent power series, such as

$$\exp(A) := I + A + \frac{1}{2}A^2 + \cdots + \frac{1}{k!}A^k + \cdots .$$

To begin, suppose that an $n \times n$ matrix A has a basis of eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$, with corresponding eigenvalues $A\mathbf{x}_i = \lambda_i \mathbf{x}_i$. (The eigenvalues are not necessarily distinct.) Then, as we showed in the previous section, we can write

$$A = X\Lambda X^{-1} = \left(\begin{array}{c|c|c} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{array} \right) \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \left(\begin{array}{c|c|c} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{array} \right)^{-1}.$$

This factorization is compatible with polynomial evaluation. That is, for any polynomial $f(x)$, I claim that

$$\boxed{f(A) = X \cdot f(\Lambda) \cdot X^{-1}.}$$

If A (hence also Λ) is invertible, then we can even allow negative powers of x in the polynomial. Such expressions are called *Laurent polynomials*:

$$f(x) = b_{-\ell}x^{-\ell} + b_{-\ell+1}x^{-\ell+1} + \cdots + b_{k-1}x^{k-1} + b_k x^k \qquad \text{for some } k, \ell \geq 0.$$

Actually, we will prove the more general fact that $A = XBX^{-1}$ implies $f(A) = X \cdot f(B) \cdot X^{-1}$ for any polynomial $f(x)$, and for any Laurent polynomial $f(x)$ when A (hence also B) is invertible. The first step is to prove that

$$\boxed{(XBX^{-1})^k = XB^kX^{-1} \text{ for all } k \geq 0, \text{ and also for } k < 0 \text{ when } B \text{ is invertible.}}$$

For this we use induction. When $k = 0$ we have $XB^0X^{-1} = XI_nX^{-1} = XX^{-1} = I_n = (XBX^{-1})^0$. Then for $k \geq 1$ we have

$$\begin{aligned} (XBX^{-1})^k &= (XBX^{-1})(XBX^{-1})^{k-1} \\ &= (XBX^{-1})(XB^{k-1}X^{-1}) && \text{induction} \end{aligned}$$

$$
\begin{aligned}
&= XB(X^{-1}X)B^{k-1}X^{-1} \\
&= XBB^{k-1}X^{-1} \\
&= XB^kX^{-1}.
\end{aligned}
$$

If B is invertible, then for all $k \geq 0$ we also have

$$(XBX^{-1})^{-k} = [(XBX^{-1})^{-1}]^k = (XB^{-1}X^{-1})^k = X(B^{-1})^kX^{-1} = XB^{-k}X^{-1}.$$

Finally, for any polynomial $f(x) = b_0 + b_1x + \cdots + b_kx^k$ we have

$$
\begin{aligned}
f(A) &= b_0I + b_1A + b_2A^2 + \cdots + b_kA^k \\
&= b_0I + b_1(XBX^{-1}) + b_2(XBX^{-1})^2 + \cdots + b_k(XBX^{-1})^k \\
&= b_0(XX^{-1}) + b_1(XBX^{-1}) + b_2(XB^2X^{-1}) + \cdots + b_k(XB^kX^{-1}) \\
&= X(b_0I + b_1B + b_2B^2 + \cdots + b_kB^k)X^{-1} \\
&= X \cdot f(B) \cdot X^{-1}.
\end{aligned}
$$

The proof for Laurent polynomials is the same.

So far, this is not very useful. It becomes useful because of the following basic observation.

Multiplication of Diagonal Matrices is Easy. The formula for a general matrix product AB is complicated. However, multiplication of diagonal matrices is easy:

$$\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \begin{pmatrix} b_1 & & \\ & \ddots & \\ & & b_n \end{pmatrix} = \begin{pmatrix} a_1b_1 & & \\ & \ddots & \\ & & a_nb_n \end{pmatrix}.$$

It follows that for any diagonal matrix Λ and any (Laurent) polynomial $f(x)$ we have

$$f(\Lambda) = f\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} = \begin{pmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{pmatrix}.$$

This is the reason why diagonalization is a powerful technique.

Here is a first application. Recall the following results from the previous two sections:

- If the characteristic polynomial $\chi_A(x)$ has distinct roots then A is diagonalizable.
- If $f(A) = O$ for some polynomial $f(x)$ with distinct roots then A is diagonalizable.

The next theorem ties these results together.

The Cayley-Hamilton Theorem. Let A be a square matrix with characteristic polynomial $\chi_A(x) = \det(xI - A)$. Then we have

$$\boxed{\chi_A(A) = O.}$$

This is a strange idea, so let's first examine the 2×2 case. Consider the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

with characteristic polynomial

$$\chi_A(\lambda) = \lambda^2 - (a+d)\lambda + (ad - bc).$$

Then one can check (as Cayley and Hamilton did) that

$$\begin{aligned} \chi_A(A) &= A^2 - (a+d)A + (ad-bc)I \\ &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 - (a+d)\begin{pmatrix} a & b \\ c & d \end{pmatrix} + (ad-bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \text{some calculations} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Why on earth should this be true? It is because of diagonalization.

Proof of Cayley-Hamilton. Suppose first that A is diagonalizable, with $A = X\Lambda X^{-1}$. For any eigenvalue λ of A, the characteristic polynomial satisfies $\chi_A(\lambda) = 0$ by definition. Hence

$$\chi_A(A) = X \cdot \chi_A(\Lambda) \cdot X^{-1} = X \begin{pmatrix} \chi_A(\lambda_1) & & \\ & \ddots & \\ & & \chi_A(\lambda_n) \end{pmatrix} X^{-1} = XOX^{-1} = O.$$

The result for non-diagonalizable matrices follows by continuity. That is, any non-diagonalizable matrix is a limit of diagonalizable matrices. And the entries of the matrix $\chi_A(A)$ are continuous functions of the entries of A. But each entry of $\chi_A(A)$ is zero for any diagonal matrix. Hence the entries of the limit are zero.[19] $\square$

Remark: The Cayley-Hamilton is actually more general than this. It holds over any commutative ring. As written, our proof only works over the complex numbers.

Next we consider two examples of infinite power series.

[19]This is a typical way to deal with non-diagonalizable matrices, i.e., view them as limits of diagonalizable matrices in the space $\mathbb{C}^{n\times n}$ of square matrices.

The Geometric Series. Consider an $n \times n$ matrix A. Suppose that A is diagonalizable with

$$A = X\Lambda X^{-1} = X \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} X^{-1}.$$

Evaluating A at the polynomial $f(x) = 1 - x$ gives

$$I - A = X(I - \Lambda)X^{-1} = X \begin{pmatrix} 1-\lambda_1 & & \\ & \ddots & \\ & & 1-\lambda_n \end{pmatrix} X^{-1}.$$

If none of the eigenvalues is 1 then $I - \Lambda$ (hence also $I - A$) is invertible, and we obtain

$$(I - A)^{-1} = X(I - \Lambda)^{-1}X^{-1} = X \begin{pmatrix} 1/(1-\lambda_1) & & \\ & \ddots & \\ & & 1/(1-\lambda_n) \end{pmatrix} X^{-1}.$$

On the other hand, for all $k \geq 0$ we can evaluate A at $f(x) = 1 + x + \cdots + x^k$ to obtain

$$\begin{aligned} I + A + \cdots + A^k &= X(I + \Lambda + \cdots + \Lambda^k)X^{-1} \\ &= X \begin{pmatrix} 1+\lambda_1+\cdots+\lambda_1^k & & \\ & \ddots & \\ & & 1+\lambda_n+\cdots+\lambda_n^k \end{pmatrix} X^{-1}. \end{aligned}$$

Finally, suppose that the eigenvalues satisfy $0 < |\lambda_i| < 1$ for all i. Then the usual geometric series for scalars implies that

$$I + \Lambda + \cdots + \Lambda^k \to (I - \Lambda)^{-1} \quad \text{as } k \to \infty.$$

The convergence is componentwise in each entry of the matrix. For a fixed invertible matrix X, the function $B \mapsto XBX^{-1}$ is continuous in the matrix entries, hence

$$X(I + \Lambda + \cdots + \Lambda^k)X^{-1} \to X(I - \Lambda)^{-1}X^{-1} \quad \text{as } k \to \infty.$$

In summary, for a diagonalizable matrix A with eigenvalues satisfying $0 < |\lambda| < 1$, we have

$$I + A + \cdots + A^k \to (I - A)^{-1} \text{ componentwise.}$$

And by continuity, the result also holds for non-diagonalizable matrices.[20] On a previous homework you proved a weaker version of this result, using more difficult techniques. Diagonalization makes things easier because it turns matrix arithmetic into scalar arithmetic.

[20] Basically, this is because the eigenvalues depend continuously on the matrix entries. I don't want to get specific about it.

The Matrix Exponential. Given a square matrix A, the functional calculus allows us to define $f(A)$ for any power series $f(x) = a_0 + a_1 x + a_2^2 + \cdots$, as long as this power series converges when evaluated at the eigenvalues of A. For example, consider the power series definition of the exponential function

$$\exp(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots .$$

It is a basic theorem of analysis that $\exp(x)$ converges for any complex number $x \in \mathbb{C}$. In order to define $\exp(A)$ we first suppose that A is diagonalizable:

$$A = X\Lambda X^{-1} = X \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} X^{-1}.$$

For any $k \geq 0$ we have

$$\sum_{i=0}^{k} \frac{1}{i!} A^i = X \left(\sum_{i=0}^{k} \frac{1}{i!} \Lambda^i \right) X^{-1} = X \begin{pmatrix} \sum_{i=0}^{k} \frac{1}{i!}\lambda_1^i & & \\ & \ddots & \\ & & \sum_{i=0}^{k} \frac{1}{i!}\lambda_n^i \end{pmatrix} X^{-1}.$$

Since the power series for $\exp(x)$ converges everywhere, we have

$$\sum_{i=1}^{k} \frac{1}{i!} \cdot \Lambda_i \to \begin{pmatrix} \exp(\lambda_1) & & \\ & \ddots & \\ & & \exp(\lambda_n) \end{pmatrix} \quad \text{as } k \to \infty.$$

Then since conjugation by the fixed matrix X is continuous, we conclude that

$$\sum_{i=1}^{k} \frac{1}{i!} \cdot A^i \to X \begin{pmatrix} \exp(\lambda_1) & & \\ & \ddots & \\ & & \exp(\lambda_n) \end{pmatrix} X^{-1} \quad \text{as } k \to \infty.$$

This establishes the existence of the matrix exponential for any diagonalizable matrix A:[21]

$$\boxed{\exp(A) = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots .}$$

Let me warn you that

$$\exp(A + B) \neq \exp(A)\exp(B) \text{ for general matrices } A, B.$$

[21] We can also prove existence for non-diagonalizable matrices using a continuity argument, though this proof doesn't tell us how to compute $\exp(A)$ in the non-diagonalizable case. The computation of $\exp(A)$ for non-diagonalizable A uses the Jordan canonical form.

The proof of $\exp(x+y) = \exp(x)\exp(y)$ relied on the fact that scalars commute. If $AB = BA$ then this same proof carries over, and we have

$$\boxed{\exp(A+B) = \exp(A)\exp(B) \text{ for matrices satisfying } AB = BA.}$$

Later we will see that the matrix exponential is the key to solving differential equations. In that context we will consider the series

$$\exp(At) = I + At + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots,$$

where t is a real variable representing time.

For now, we present two example computations. First consider the matrix

$$A = \frac{1}{6}\begin{pmatrix} 5 & 4 \\ 2 & -2 \end{pmatrix}.$$

The characteristic polynomial is

$$\begin{aligned} \det(xI - A) &= (x - 5/6)(x + 2/6) - (-4/6)(-2/6) \\ &= x^2 - (1/2)x - (1/2) \\ &= (x-1)(x+1/2), \end{aligned}$$

hence the eigenvalues are 1 and $-1/2$. Since this 2×2 matrix has 2 distinct eigenvalues, we know that it is diagonalizable. After some computation we find the eigenvectors:

$$A\begin{pmatrix} 4 \\ 1 \end{pmatrix} = 1\begin{pmatrix} 4 \\ 1 \end{pmatrix} \quad \text{and} \quad A\begin{pmatrix} 1 \\ -2 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Hence we obtain the diagonalization:

$$A = \left(\begin{array}{c|c} 4 & 1 \\ 1 & -2 \end{array}\right)\begin{pmatrix} 1 & 0 \\ 0 & -1/2 \end{pmatrix}\left(\begin{array}{c|c} 4 & 1 \\ 1 & -2 \end{array}\right)^{-1} = \frac{1}{9}\begin{pmatrix} 4 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1/2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -4 \end{pmatrix}.$$

Finally, we obtain the exponential:

$$\exp(A) = \frac{1}{9}\begin{pmatrix} 4 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} \exp(1) & 0 \\ 0 & \exp(-1/2) \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -4 \end{pmatrix}.$$

The last example is more interesting. Consider the matrix that rotates by 90°:

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

For any real number θ, we will show that

$$\exp(R\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos t \end{pmatrix} = \cos\theta \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sin\theta \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

which is the matrix that rotates by θ. This is a matrix version of Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

where the 90° rotation matrix R plays the role of the imaginary unit i. We begin by computing the eigenvalues. The characteristic polynomial is

$$\det(xI - R) = \det\begin{pmatrix} x & 1 \\ -1 & x \end{pmatrix} = x^2 + 1,$$

hence there are two distinct eigenvalues: i and $-i$. It is no surprise that these are complex conjugates, since the complex eigenvalues of real matrices come in conjugate pairs. (See the homework.) With a bit of work, one finds the eigenvectors

$$R\begin{pmatrix} 1 \\ -i \end{pmatrix} = i\begin{pmatrix} 1 \\ -i \end{pmatrix} \quad \text{and} \quad R\begin{pmatrix} 1 \\ i \end{pmatrix} = -i\begin{pmatrix} 1 \\ i \end{pmatrix},$$

and hence the exponential:

$$\exp(R\theta) = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}\begin{pmatrix} \exp(i\theta) & 0 \\ 0 & \exp(-i\theta) \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}^{-1}.$$

Then some simplification using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ gives the desired result.

But this makes the result look like a miracle. We can gain more insight by looking at the real and imaginary parts of the complex eigenvalues. Let $\mathbf{x} = (1, -i)$, so that $\mathbf{x} = (1, 0) - i(0, 1)$. Since $R\mathbf{x} = i\mathbf{x}$ we must also have $\exp(R\theta)\mathbf{x} = \exp(i\theta)\mathbf{x}$,[22] and hence

$$\begin{aligned}
\exp(R\theta)\begin{pmatrix} 1 \\ 0 \end{pmatrix} - i\exp(R\theta)\begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \exp(R\theta)\begin{pmatrix} 1 \\ -i \end{pmatrix} \\
&= \exp(i\theta)\begin{pmatrix} 1 \\ -i \end{pmatrix} \\
&= (\cos\theta + i\sin\theta)\begin{pmatrix} 1 \\ -i \end{pmatrix} && \text{Euler's formula} \\
&= \begin{pmatrix} \cos\theta + i\sin\theta \\ \sin\theta - i\cos\theta \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} - i\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}.
\end{aligned}$$

Since the matrix $\exp(R\theta)$ has real entries, comparing real and imaginary parts gives

$$\exp(R\theta)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \quad \text{and} \quad \exp(R\theta)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix},$$

which is the desired result.

Remark: In general, complex eigenvalues of real matrices lead to rotation. We will examine this in the next section.

[22] The proof that $A\mathbf{x} = \lambda\mathbf{x}$ implies $f(A)\mathbf{x} = f(\lambda)\mathbf{x}$ for polynomials $f(x)$ carries over to power series.

1.6 Complex Eigenvalues and Eigenvectors of Real Matrices

For any complex number $a + ib \in \mathbb{C}$ with $a, b \in \mathbb{R}$ we will denote its complex conjugate by

$$\overline{\alpha} = \overline{a + ib} = a - ib.$$

Recall that complex conjugation satisfies the following properties:

- $\overline{\alpha + \beta} = \overline{\alpha} + \overline{\beta}$,
- $\overline{\alpha\beta} = \overline{\alpha} \cdot \overline{\beta}$,
- $\overline{\alpha^k} = (\overline{\alpha})^k$,
- $\overline{\alpha} = \alpha$ if and only if $\alpha \in \mathbb{R}$.

Given a polynomial $f(x) = b_0 + b_1 x + \cdots + b_n x^n$ with **real coefficients** and a complex number $\alpha \in \mathbb{C}$, it follows from these properties that

$$\begin{aligned}\overline{f(\alpha)} &= \overline{b_0 + b_1\alpha + \cdots + b_n\alpha^n} \\ &= \overline{b_0} + \overline{b_1} \cdot \overline{\alpha} + \cdots + \overline{b_n} \cdot \overline{\alpha^n} \\ &= b_0 + b_1\overline{\alpha} + \cdots + b_n(\overline{\alpha})^n \\ &= f(\overline{\alpha}).\end{aligned}$$

In particular, we see that α is a root of $f(x)$ if and only if $\overline{\alpha}$ is a root of $f(x)$. Indeed, if $f(\alpha) = 0$ then

$$f(\overline{\alpha}) = \overline{f(\alpha)} = \overline{0} = 0,$$

and if $f(\overline{\alpha}) = 0$ then

$$f(\alpha) = \overline{\overline{f(\alpha)}} = \overline{f(\overline{\alpha})} = \overline{0} = 0.$$

It follows from this that

$$\boxed{\text{the non-real roots of a real polynomial come in conjugate pairs.}}$$

And, as an interesting consequence,

$$\boxed{\text{every real polynomial of odd degree has as least one real root.}}$$

We will apply these observations to eigenvalues of real matrices.

Complex Eigenvalues of a Real Matrix. Let A be an $n \times n$ matrix with **real entries**, so the characteristic polynomial $\chi_A(x) = \det(xI - A)$, has **real coefficients**. According to the previous result, the characteristic polynomial can be factored as

$$\chi_A(x) = (x - \lambda_1) \cdots (x - \lambda_{n-2m})(x - \alpha_1)(x - \overline{\alpha_1}) \cdots (x - \alpha_m)(x - \overline{\alpha_m}),$$

for some real numbers $\lambda_i \in \mathbb{R}$ and non-real complex numbers $\alpha_i \in \mathbb{C}$. If n is even, then the matrix A need not have any real eigenvalues. For example, the real matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

has characteristic polynomial $(x-i)(x+i)$. On the other hand, if n is odd then the number $n-2m$ must be ≥ 1, so that A has at least one real eigenvalue.

Complex Eigenvectors. If a real matrix A has a real eigenvalue λ, then the corresponding eigenvectors are real.[23] Indeed, the space of λ-eigenvectors is the null space $\mathcal{N}(\lambda I - A)$, which can be computed by elimination over $\mathbb{R}$. What about complex eigenvalues? Suppose that a real $n \times n$ matrix A has a complex eigenvalue $\lambda \in \mathbb{C}$, and let $\mathbf{x} \in \mathbb{C}^n$ be a corresponding eigenvector:

$$A\mathbf{x} = \lambda\mathbf{x}.$$

If λ is not real then $\mathbf{x}$ cannot have real entries. Indeed, if $\mathbf{x} \in \mathbb{R}^n$ then since A has real entries we would have $\lambda\mathbf{x} = A\mathbf{x} \in \mathbb{R}^n$ which implies that $\lambda \in \mathbb{R}$. Let us suppose that $\lambda = a + ib$ with $a, b \in \mathbb{R}$ and $b \neq 0$. Then we can write

$$\mathbf{x} = \mathbf{u} + i\mathbf{v}$$

for unique real vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ with $\mathbf{v} \neq \mathbf{0}$. By expanding the equation $A\mathbf{x} = \lambda\mathbf{x}$ we obtain

$$\begin{aligned} A\mathbf{u} + iA\mathbf{v} &= A(\mathbf{u} + i\mathbf{v}) \\ &= A\mathbf{x} \\ &= \lambda\mathbf{x} \\ &= (a+ib)(\mathbf{u} + i\mathbf{v}) \\ &= (a\mathbf{u} - b\mathbf{v}) + i(b\mathbf{u} + a\mathbf{v}). \end{aligned}$$

Since the vectors $A\mathbf{u}$, $A\mathbf{v}$, $a\mathbf{u} - \mathbf{bv}$ and $b\mathbf{u} + a\mathbf{v}$ have real entries, it follows by comparing real and imaginary parts that

$$\begin{cases} A\mathbf{u} &= a\mathbf{u} - b\mathbf{v}, \\ A\mathbf{v} &= b\mathbf{u} + a\mathbf{v}, \end{cases}$$

which can be expressed as a matrix equation:

$$A\left(\mathbf{u} \mid \mathbf{v} \right) = \left(\mathbf{u} \mid \mathbf{v} \right) \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

Next, I claim that the vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly independent over $\mathbb{C}$.[24] To see this, we note that the conjugate vector $\overline{\mathbf{x}} = \mathbf{u} - i\mathbf{v}$ is an eigenvector of A corresponding to the conjugate eigenvalue $\overline{\lambda} = a - ib$. Indeed, since A has real entries, conjugating both sides of the equation $A\mathbf{x} = \lambda\mathbf{x}$ gives[25]

$$A\overline{\mathbf{x}} = \overline{A}\overline{\mathbf{x}} = \overline{A\mathbf{x}} = \overline{\lambda\mathbf{x}} = \overline{\lambda}\overline{\mathbf{x}}.$$

Since $\lambda \neq \overline{\lambda}$, the vectors $\mathbf{x}, \overline{\mathbf{x}} \in \mathbb{C}$ correspond to different eigenvalues, hence they are linearly independent over $\mathbb{C}$. But then since

$$\left(\mathbf{x} \mid \overline{\mathbf{x}} \right) = \left(\mathbf{u} \mid \mathbf{v} \right) \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \quad \text{where } \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \text{ is invertible,}$$

[23] Technically, every λ-eigenvector is a scalar multiple of a real vector. You could take a real λ-eigenvector $\mathbf{x}$ and scale it to get a complex λ-eigenvector $i\mathbf{x}$, but why would you want to do that?

[24] In this section we will only discuss linear independence over $\mathbb{C}$, which implies linear independence over $\mathbb{R}$.

[25] The equation $\overline{A\mathbf{x}} = \overline{A}\overline{\mathbf{x}}$ needs to be checked. It follows from the standard properties.

we conclude that $\mathbf{u}$ and $\mathbf{v}$ are linearly independent. In particular, we have

$$A = \left(\ \mathbf{u} \ \middle| \ \mathbf{v} \ \right) \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \left(\ \mathbf{u} \ \middle| \ \mathbf{v} \ \right)^{-1}.$$

Furthermore, for any $\mathbf{y} \in \mathbb{C}^n$ we have

$$\left(\ \mathbf{y} \ \middle| \ \mathbf{x} \ \middle| \ \overline{\mathbf{x}} \ \right) = \left(\ \mathbf{y} \ \middle| \ \mathbf{u} \ \middle| \ \mathbf{v} \ \right) \begin{pmatrix} 1 & & \\ & 1 & 1 \\ & i & -i \end{pmatrix},$$

which implies that the set $\mathbf{y}, \mathbf{x}, \overline{\mathbf{x}}$ is independent if and only if $\mathbf{y}, \mathbf{u}, \mathbf{v}$ is independent.

Real Diagonalizable Matrices. Finally, we discuss diagonalization of real matrices. Let A be a real $n \times n$ matrix. As we saw above, the characteristic polynomial can be factored as

$$\chi_A(x) = (x - \lambda_1) \cdots (x - \lambda_{n-2m})(x - \alpha_1)(x - \overline{\alpha_1}) \cdots (x - \alpha_m)(x - \overline{\alpha_m}),$$

where $\lambda_1, \ldots, \lambda_{n-2m}$ are real and $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$ are non-real.

Suppose that A is diagonalizable over $\mathbb{C}$. This means that we can find nonzero vectors $\mathbf{y}_1, \ldots, \mathbf{y}_{n-2m} \in \mathbb{C}^n$ and $\mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathbb{C}^n$ such that $A\mathbf{y}_i = \lambda_i \mathbf{y}_i$ and $A\mathbf{x}_i = \alpha_i \mathbf{x}_i$, hence also $A\overline{\mathbf{x}_i} = \overline{\alpha_i \mathbf{x}_i}$, and such that

$$\mathbf{y}_1, \ldots, \mathbf{y}_{n-2m}, \mathbf{x}_1, \overline{\mathbf{x}_1}, \ldots, \mathbf{x}_m, \overline{\mathbf{x}_m}$$

is a basis for $\mathbb{C}^n$. If X is the $n \times n$ (invertible) matrix with these column vectors, then we have

$$A = X \begin{pmatrix} \lambda_1 & & & & & & & \\ & \ddots & & & & & & \\ & & \lambda_{n-2m} & & & & & \\ & & & \alpha_1 & & & & \\ & & & & \overline{\alpha_1} & & & \\ & & & & & \ddots & & \\ & & & & & & \alpha_m & \\ & & & & & & & \overline{\alpha_m} \end{pmatrix} X^{-1}.$$

Now we will eliminate the complex numbers from this factorization. Since the eigenvalues $\lambda_1, \ldots, \lambda_{n-2m}$ are real, we can choose the eignevectors $\mathbf{y}_1, \ldots, \mathbf{y}_{n-2m}$ to be real. Next we write $\mathbf{x}_i = \mathbf{u}_i + i\mathbf{v}_i$ for real vectors $\mathbf{u}_i, \mathbf{v}_i$ with $\mathbf{v}_i \neq \mathbf{0}$. From the previous remarks we see that

$$\mathbf{y}_1, \ldots, \mathbf{y}_{n-2m}, \mathbf{u}_1, \mathbf{v}_1, \ldots, \mathbf{u}_m, \mathbf{v}_m$$

is a basis for $\mathbb{C}^n$ consisting of real vectors. Furthermore, if Y is the (invertible) matrix with these columns, then we have

$$A = Y \begin{pmatrix} \lambda_1 & & & & & & & \\ & \ddots & & & & & & \\ & & \lambda_{n-2m} & & & & & \\ & & & a_1 & b_1 & & & \\ & & & -b_1 & a_1 & & & \\ & & & & & \ddots & & \\ & & & & & & a_m & b_m \\ & & & & & & -b_m & a_m \end{pmatrix} Y^{-1}.$$

This is not quite a “diagonalization”, but it has the virtue using only real numbers.

1.7 Normal Matrices

In this chapter we have studied the evaluation of polynomials (also power series and Laurent polynomials) at matrices. This discussion has left out one important operation; namely, the transpose and conjugate transpose. In this final section we consider the relationship between eigenvalues and (conjugate) transposition.

The main role is played by *normal matrices*. We say that a matrix A is normal when it commutes with its (conjugate) transpose:

$$A^*A = AA^*.$$

These matrices are extremely common in applications and include the following four families:

- Real symmetric matrices $A^T = A$.
- Complex Hermitian matrices $A^* = A$.
- Real orthogonal matrices $A^T = A^{-1}$.
- Complex unitary matrices $A^* = A^{-1}$.

Of course, these families could be dealt with separately. The reason to combine them under the concept of normal matrices is because of the following fundamental theorem, which we will prove in the next chapter.

The Spectral Theorem. Let A be a square matrix over $\mathbb{R}$ or $\mathbb{C}$. Then

$$A^*A = AA^* \quad \Longleftrightarrow \quad A \text{ has an orthonormal basis of eigenvectors.}$$

Actually, some people think it undignified to call this the Spectral Theorem. They say that the true Spectral Theorem applies to operators on infinite dimensional Hilbert spaces. Recall, if V is a real or complex Hilbert space and if $A : V \to V$ is a bounded[26] linear operator then there exists a unique bounded linear operator $A^* : V \to V$ satisfying

$$\langle A\mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{x}, A^*\mathbf{y}\rangle \quad \text{for all } \mathbf{x}, \mathbf{y} \in V.$$

As with many results in functional analysis, the proof is 80% algebra and 20% analysis, which is mostly plausible from geometric intuition.

Anyway, it is convenient to state and prove the results of this section in a language that applies also to Hilbert spaces. Our first theorem was proved by Cauchy in 1829, as part of his extension of the Principal Axes Theorem to higher dimensions. Cauchy's original proof was quite complicated, but today's proof is a one-liner.[27]

Cauchy's Reality Theorem. A real symmetric matrix has real eigenvalues.

Actually, we will prove the following more general statement, since it has the same proof.

Theorem. A self-adjoint operator on a complex inner product space has real eigenvalues.

Proof. Let V be a real or complex inner product space and let $A : V \to V$ be an operator satisfying $A^* = A$. If $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ and nonzero vector $\mathbf{x} \neq \mathbf{0}$ then we have

$$\begin{aligned}
\lambda\|\mathbf{x}\|^2 &= \lambda\langle \mathbf{x}, \mathbf{x}\rangle \\
&= \langle \mathbf{x}, \lambda\mathbf{x}\rangle \\
&= \langle \mathbf{x}, A\mathbf{x}\rangle \\
&= \langle A^*\mathbf{x}, \mathbf{x}\rangle \\
&= \langle A\mathbf{x}, \mathbf{x}\rangle && A^* = A \\
&= \langle \lambda\mathbf{x}, \mathbf{x}\rangle \\
&= \overline{\lambda}\langle \mathbf{x}, \mathbf{x}\rangle \\
&= \overline{\lambda}\|\mathbf{x}\|^2.
\end{aligned}$$

Since $\|\mathbf{x}\| \neq 0$ this implies that $\overline{\lambda} = \lambda$, and hence λ is real. $\square$

The next theorem has a similar proof.

Theorem. Unitary (and real orthogonal) operators have eigenvalues of length 1. That is, they have eigenvalues of the form $e^{i\theta}$.

Proof. Let V be a real or complex inner product space and let $A : V \to V$ be an operator satisfying $A^*A = I$. If $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ and nonzero vector $\mathbf{x} \neq \mathbf{0}$ then we have

$$\|\mathbf{x}\|^2 = \langle \mathbf{x}, \mathbf{x}\rangle$$

[26] This means that A sends the unit ball to a bounded set. It is equivalent to A being continuous.

[27] "Dazzled by the brilliance of the new theory of determinants, mathematicians overlooked simple inner product considerations", Hawkins, *The Mathematics of Frobenius in Context*, page 98.

$$
\begin{aligned}
&= \langle \mathbf{x}, I\mathbf{x} \rangle \\
&= \langle \mathbf{x}, A^*A\mathbf{x} \rangle && A^*A = I \\
&= \langle A\mathbf{x}, A\mathbf{x} \rangle \\
&= \langle \lambda\mathbf{x}, \lambda\mathbf{x} \rangle \\
&= \lambda \langle \lambda\mathbf{x}, \mathbf{x} \rangle \\
&= \overline{\lambda}\lambda \langle \mathbf{x}, \mathbf{x} \rangle \\
&= |\lambda|^2 \|\mathbf{x}\|^2.
\end{aligned}
$$

Since $\|\mathbf{x}\| \neq 0$ this implies that $|\lambda| = 1$. $\square$

Though it doesn't involve eigenvalues, we should probably include the following result.

Theorem. Unitary operators preserve lengths and angles.

Proof. This follows from the fact that unitary operators preserve the inner product. If $A^*A = I$ then for all vectors $\mathbf{x}, \mathbf{y}$ we have

$$\langle A\mathbf{x}, A\mathbf{y} \rangle = \langle \mathbf{x}, A^*A\mathbf{y} \rangle = \langle \mathbf{x}, I\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle.$$

$\square$

We have seen that the important families of normal matrices have quite restricted eigenvalues:

- Real symmetric matrices $A^T = A$ and complex Hermitian matrices $A^* = A$ matrices have real eigenvalues.
- Real orthogonal matrices $A^T A = I$ and complex unitary matrices $A^*A = I$ have eigenvalues of the form $e^{i\theta}$.

On the other hand, a general normal matrix can have any eigenvalues you want. Indeed, consider any complex numbers $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and let Λ be the diagonal matrix with these numbers on the diagonal. Then for any unitary matrix $U^*U = I$, the matrix

$$A = U\Lambda U^*$$

is normal and has eigenvalues $\lambda_1, \ldots, \lambda_n$.

What about eigen**vectors**? In this case, the key property is shared by all normal operators. This result is a precursor to the Spectral Theorem.

Theorem (Normal with Distinct Eigenvalues $\Rightarrow$ Orthogonal Eigenvectors). Let $A^*A = AA^*$ be a normal operator on an inner product space. Then

$$A\mathbf{x} = \lambda\mathbf{x} \text{ and } A\mathbf{y} = \mu\mathbf{y} \text{ with } \lambda \neq \mu \quad \Longrightarrow \quad \langle \mathbf{x}, \mathbf{y} \rangle = 0.$$

We will work up to the proof by a series of lemmas.

Lemma 1. If $A^*A = AA^*$ then $\langle A\mathbf{x}, A\mathbf{y}\rangle = \langle A^*\mathbf{x}, A^*\mathbf{y}\rangle$ for all $\mathbf{x}, \mathbf{y}$.

Proof. We have $\langle A\mathbf{x}, A\mathbf{y}\rangle = \langle \mathbf{x}, A^*A\mathbf{y}\rangle = \langle \mathbf{x}, AA^*\mathbf{y}\rangle = \langle A^*\mathbf{x}, A^*\mathbf{y}\rangle$. $\square$

Lemma 2. If $A^*A = AA^*$ then we have $A\mathbf{x} = 0$ if and only if $A^*\mathbf{x} = \mathbf{0}$.

Proof. Putting $\mathbf{y} = \mathbf{x}$ in Lemma 1 gives $\|A\mathbf{x}\|^2 = \langle A\mathbf{x}, A\mathbf{x}\rangle = \langle A^*\mathbf{x}, A^*\mathbf{x}\rangle = \|A^*\mathbf{x}\|^2$. Hence

$$A\mathbf{x} = \mathbf{0} \quad \Longleftrightarrow \quad \|A\mathbf{x}\| = 0 \quad \Longleftrightarrow \quad \|A^*\mathbf{x}\| = 0 \quad \Longleftrightarrow \quad A^*\mathbf{x} = \mathbf{0}.$$

$\square$

Lemma 3. Let $A^*A = AA^*$. Then for any vector $\mathbf{x}$ and scalar λ we have

$$A\mathbf{x} = \lambda\mathbf{x} \quad \Longleftrightarrow \quad A^*\mathbf{x} = \overline{\lambda}\mathbf{x}.$$

Proof. Consider the matrix $B = \lambda I - A$, with $B^* = \overline{\lambda} I - A^*$. Then B is normal:

$$\begin{aligned}
B^*B &= (\overline{\lambda} I - A^*)(\lambda I - A) \\
&= \overline{\lambda}\lambda I - \overline{\lambda} A - \lambda A^* + A^*A \\
&= \overline{\lambda}\lambda I - \overline{\lambda} A - \lambda A^* + AA^* && A^*A = AA^* \\
&= (\lambda I - A)(\overline{\lambda} I - A^*) \\
&= BB^*.
\end{aligned}$$

Hence applying Lemma 2 gives

$$A\mathbf{x} = \lambda\mathbf{x} \quad \Longleftrightarrow \quad B\mathbf{x} = \mathbf{0} \quad \Longleftrightarrow \quad B^*\mathbf{x} = \mathbf{0} \quad \Longleftrightarrow \quad A^*\mathbf{x} = \overline{\lambda}\mathbf{x}.$$

Proof of the Theorem. Let $A^*A = AA^*$ and suppose that $A\mathbf{x} = \lambda\mathbf{x}$ and $A\mathbf{y} = \mu\mathbf{y}$ with $\lambda \neq \mu$. Then from Lemma 3 we have $A^*\mathbf{x} = \overline{\lambda}\mathbf{x}$, hence

$$\begin{aligned}
\lambda\langle \mathbf{x}, \mathbf{y}\rangle &= \langle \overline{\lambda}\mathbf{x}, \mathbf{y}\rangle \\
&= \langle A^*\mathbf{x}, \mathbf{y}\rangle && \text{Lemma 3} \\
&= \langle \mathbf{x}, A\mathbf{y}\rangle \\
&= \langle \mathbf{x}, \mu\mathbf{y}\rangle \\
&= \mu\langle \mathbf{x}, \mathbf{y}\rangle.
\end{aligned}$$

Finally, since $(\lambda - \mu)\langle \mathbf{x}, \mathbf{y}\rangle$ and $\lambda \neq \mu$ we have $\langle \mathbf{x}, \mathbf{y}\rangle$. $\square$

In particular, this shows that an $n \times n$ normal matrix $A^*A = AA^*$ with n distinct eigenvalues has an orthogonal basis of eigenvectors. The Spectral Theorem says that this is still true even if A has repeated eigenvalues. The hard part is to show that there are **enough** eigenvectors to fill up the whole space. See the next chapter.

2 Factorization Theorems

2.1 Gram-Schmidt and QR Factorization

Most of the theorems in this chapter deal with orthonormal bases. In this section we lay the groundwork by showing how any basis can be converted into an orthonormal basis. The procedure is quite general. First we consider an infinite dimensional inner product space V over $\mathbb{R}$ or $\mathbb{C}$. Afterwards we will consider finite dimensional spaces and matrices.

Given any linearly independent set $\mathbf{a}_1, \mathbf{a}_2, \ldots \in V$ the Gram-Schmidt procedure produces linearly independent vectors $\mathbf{b}_1, \mathbf{b}_2, \ldots \in V$ with the following properties:[28]

- $\langle \mathbf{b}_i, \mathbf{b}_j \rangle = 0$ for $i \neq j$,
- $\mathrm{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_k\} = \mathrm{Span}\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$.

The definition is recursive:

- First set $\mathbf{b}_1 := \mathbf{a}_1$.
- Then for any $k \geq 1$ set $\mathbf{b}_{k+1} := \mathbf{a}_{k+1} - \mathrm{Proj}_k(\mathbf{a}_{k+1})$, where $\mathrm{Proj}_k : V \to V$ is the orthogonal projection onto the subspace spanned by $\mathbf{b}_1, \ldots, \mathbf{b}_k$. To be precise, we set

$$\mathbf{b}_{k+1} := \mathbf{a}_{k+1} - \frac{\langle \mathbf{a}_{k+1}, \mathbf{b}_1 \rangle}{\langle \mathbf{b}_1, \mathbf{b}_1 \rangle}\mathbf{b}_1 - \cdots - \frac{\langle \mathbf{a}_{k+1}, \mathbf{b}_k \rangle}{\langle \mathbf{b}_k, \mathbf{b}_k \rangle}\mathbf{b}_k.$$

You will prove on the homework that this procedure has the desired properties. Afterwards, we can easily turn $\mathbf{b}_1, \mathbf{b}_2, \ldots \in V$ into an ortho**normal** set by dividing each $\mathbf{b}_k$ by its length.

Before applying this to matrices, we give one application to infinite dimensional function spaces. Consider the real Hilbert space $L^2[-1,1]$ with inner product

$$\langle f(x), g(x) \rangle = \int_{-1}^{1} f(x)g(x)\,dx.$$

And consider the "obvious" basis $1, x, x^2, \ldots \in L^2[-1,1]$.[29] Note that these functions are not orthogonal. For example,

$$\langle 1, x^2 \rangle = \int_{-1}^{1} x^2\,dx = \frac{1}{3}x^3\Big|_{-1}^{1} = \frac{1}{3}(1)^3 - \frac{1}{3}(-1)^3 = \frac{1}{3} + \frac{1}{3} = \frac{2}{3} \neq 0.$$

Applying the Gram-Schmidt procedure to the non-orthogonal basis $1, x, x^2, \ldots$ produces the orthogonal basis of *Legendre polynomials*: $P_0(x), P_1(x), P_2(x), \ldots$. These are used in physics in the study of spherically symmetric potentials. For example, they determine the "shapes"

[28] If $\mathbf{a}_1, \mathbf{a}_2, \ldots$ is a Hilbert space basis, with appropriate convergence properties, then the vectors $\mathbf{b}_1, \mathbf{b}_2, \ldots$ will also be a Hilbert space basis, though we won't prove this.

[29] It can be shown that this is, indeed, a Hilbert space basis.

of electron orbitals. To be precise, we first define the *associated Legendre function* for integers $\ell, m \in \mathbb{Z}$ with $0 \le m \le \ell$:

$$P_\ell^m(x) = (1-x^2)^{m/2} \cdot \frac{d^m}{dx^m} P_\ell(x).$$

Then the radial equation for the shape of the (ℓ, m)-orbital is[30]

$$\rho = (\text{constant}) \cdot |P_\ell^m(\cos\theta)|.$$

Now we turn to matrices. The matrix form of Gram-Schmidt is called QR factorization. Given an invertible $n \times n$ matrix A, we will produce a unitary matrix $Q^*Q = I$ and an upper-triangular matrix R such that $A = QR$. If A has real entries then Q and R will have real entries. In this case $Q^TQ = I$ is real orthogonal.

Let $\mathbf{a}_1, \ldots, \mathbf{a}_n$ be a basis for $\mathbb{C}^n$. Then the Gram-Schmidt basis $\mathbf{b}_1, \ldots, \mathbf{b}_n$ satisfies

$$\begin{aligned}
\mathbf{a}_1 &= \mathbf{b}_1, \\
\mathbf{a}_2 &= \mathbf{b}_2 + \frac{\langle \mathbf{a}_2, \mathbf{b}_1 \rangle}{\langle \mathbf{b}_1, \mathbf{b}_1 \rangle} \mathbf{b}_1, \\
&\vdots \\
\mathbf{a}_n &= \mathbf{b}_n + \frac{\langle \mathbf{a}_n, \mathbf{b}_{n-1} \rangle}{\langle \mathbf{b}_{n-1}, \mathbf{b}_{n-1} \rangle} \mathbf{b}_{n-1} + \cdots + \frac{\langle \mathbf{a}_n, \mathbf{b}_1 \rangle}{\langle \mathbf{b}_1, \mathbf{b}_1 \rangle} \mathbf{b}_1,
\end{aligned}$$

which can be expressed as a matrix equation:

$$A = BU$$

$$\left(\mathbf{a}_1 \,\middle|\, \cdots \,\middle|\, \mathbf{a}_n \right) = \left(\mathbf{b}_1 \,\middle|\, \cdots \,\middle|\, \mathbf{b}_n \right) \begin{pmatrix} 1 & \frac{\langle \mathbf{a}_2, \mathbf{b}_1 \rangle}{\langle \mathbf{b}_1, \mathbf{b}_1 \rangle} & \cdots & \cdots & \frac{\langle \mathbf{a}_n, \mathbf{b}_1 \rangle}{\langle \mathbf{b}_1, \mathbf{b}_1 \rangle} \\ & 1 & & & \vdots \\ & & \ddots & & \vdots \\ & & & 1 & \frac{\langle \mathbf{a}_n, \mathbf{b}_{n-1} \rangle}{\langle \mathbf{b}_{n-1}, \mathbf{b}_{n-1} \rangle} \\ & & & & 1 \end{pmatrix}$$

By construction, the columns of B are orthogonal. We can make them ortho**normal** by scaling the kth columns $\mathbf{b}_k$ by $1/\|\mathbf{b}_k\|$. If S is the diagonal matrix with entries $1/\|\mathbf{b}_k\|$, then the matrix $Q = BS$ has orthonormal columns $\mathbf{q}_k = \mathbf{b}_k/\|\mathbf{b}_k\|$, hence $Q^*Q = I$. To convert $A = BU$ into $A = QR$ we define $R = S^{-1}U$ so that

$$\begin{aligned}
A &= BU \\
&= B(SS^{-1})U \\
&= (BS)(S^{-1}U) \\
&= QR.
\end{aligned}$$

[30] This example is just for fun. See Griffiths, *Introduction to Quantum Mechanics*, Equation 4.32.

It turns out that the matrix $R = S^{-1}U$ has a nice form. To see this, we first observe that

$$\langle \mathbf{a}_k, \mathbf{b}_k \rangle = \langle \mathbf{b}_k + \text{stuff orthogonal to } \mathbf{b}_k, \mathbf{b}_k \rangle = \langle \mathbf{b}_k, \mathbf{b}_k \rangle = \|\mathbf{b}_k\|^2,$$

which implies that

$$\langle \mathbf{a}_k, \mathbf{q}_k \rangle = \left\langle \mathbf{a}_k, \frac{\mathbf{b}_k}{\|\mathbf{b}_k\|} \right\rangle = \frac{1}{\|\mathbf{b}_k\|} \langle \mathbf{a}_k, \mathbf{b}_k \rangle = \frac{1}{\|\mathbf{b}_k\|} \|\mathbf{b}_k\|^2 = \|\mathbf{b}_i\|.$$

Furthermore, for any $1 \le i < k$ we have

$$\|\mathbf{b}_i\| \cdot \frac{\langle \mathbf{a}_k, \mathbf{b}_i \rangle}{\langle \mathbf{b}_i, \mathbf{b}_i \rangle} = \|\mathbf{b}_i\| \cdot \frac{\langle \mathbf{a}_k, \mathbf{b}_i \rangle}{\|\mathbf{b}_i\|^2} = \frac{1}{\|\mathbf{b}_i\|} \cdot \langle \mathbf{a}_k, \mathbf{b}_i \rangle = \left\langle \mathbf{a}_k, \frac{\mathbf{b}_i}{\|\mathbf{b}_i\|} \right\rangle = \langle \mathbf{a}_k, \mathbf{q}_i \rangle.$$

Putting these together gives

$$\begin{aligned} R &= S^{-1}U \\ &= \begin{pmatrix} \|\mathbf{b}_1\| & & \\ & \ddots & \\ & & \|\mathbf{b}_n\| \end{pmatrix} \begin{pmatrix} 1 & \frac{\langle \mathbf{a}_2, \mathbf{b}_1 \rangle}{\langle \mathbf{b}_1, \mathbf{b}_1 \rangle} & \cdots & \cdots & \frac{\langle \mathbf{a}_n, \mathbf{b}_1 \rangle}{\langle \mathbf{b}_1, \mathbf{b}_1 \rangle} \\ & 1 & & & \vdots \\ & & \ddots & & \vdots \\ & & & 1 & \frac{\langle \mathbf{a}_n, \mathbf{b}_{n-1} \rangle}{\langle \mathbf{b}_{n-1}, \mathbf{b}_{n-1} \rangle} \\ & & & & 1 \end{pmatrix} \\ &= \begin{pmatrix} \langle \mathbf{a}_1, \mathbf{q}_1 \rangle & \cdots & \langle \mathbf{a}_n, \mathbf{q}_1 \rangle \\ & \ddots & \vdots \\ & & \langle \mathbf{a}_n, \mathbf{q}_n \rangle \end{pmatrix}. \end{aligned}$$

In summary, for any $n \times n$ invertible matrix A with columns $\mathbf{a}_i$, we can find an $n \times n$ unitary matrix $Q^*Q = I$ with columns $\mathbf{q}_i$, such that

$$\begin{aligned} \left(\ \mathbf{a}_1 \ \middle| \ \cdots \ \middle| \ \mathbf{a}_n \ \right) &= \left(\ \mathbf{q}_1 \ \middle| \ \cdots \ \middle| \ \mathbf{q}_n \ \right) \begin{pmatrix} \langle \mathbf{a}_1, \mathbf{q}_1 \rangle & \cdots & \langle \mathbf{a}_n, \mathbf{q}_1 \rangle \\ & \ddots & \vdots \\ & & \langle \mathbf{a}_n, \mathbf{q}_n \rangle \end{pmatrix} \\ A &= QR. \end{aligned}$$

And if A is real then we can choose Q and R with real entries.

Due to rounding errors, the matrix Q computed from Gram-Schmidt is only approximately orthogonal. It is worth mentioning another method, due to Householder, that gives an exactly orthogonal matrix. This method also has an interesting theoretical consequence:

$$\boxed{\text{Any real orthogonal matrix } A^TA = I \text{ is a composition of reflections.}}$$

This method uses the *Householder reflection matrices*:

$$H_{\mathbf{v}} = I - 2\frac{\mathbf{v}\mathbf{v}^T}{\|\mathbf{v}\|^2} \quad \text{for } \mathbf{v} \in \mathbb{R}^n.$$

Recall that

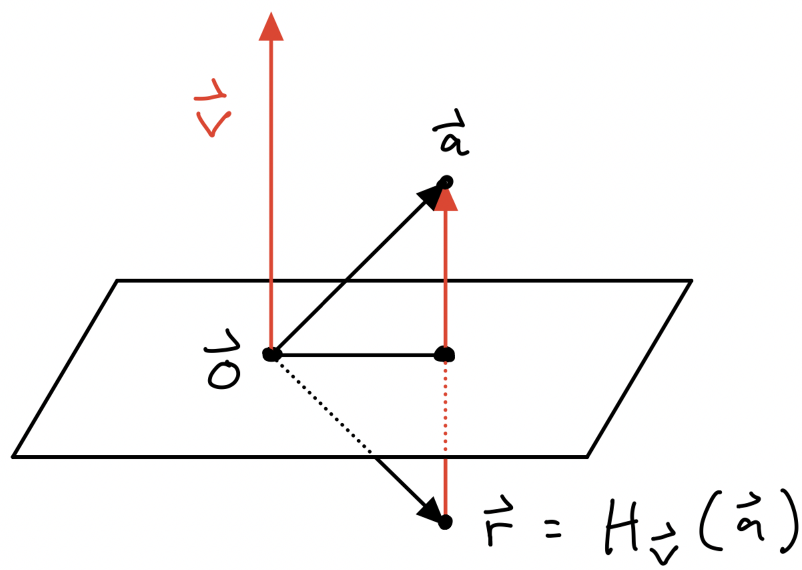
$$P_{\mathbf{v}} = \mathbf{v}(\mathbf{v}\mathbf{v}^T)^{-1}\mathbf{v}^T = \frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}} = \frac{\mathbf{v}\mathbf{v}^T}{\|\mathbf{v}\|^2}$$

is the matrix that projects orthogonally onto the line spanned by $\mathbf{v}$, hence $I - P_{\mathbf{v}}$ is the matrix that projects onto the orthogonal hyperplane $\mathbf{v}^{\perp} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^T\mathbf{v} = 0\}$. From the remarks in the previous chapter, this implies that $2P_{\mathbf{v}} - I$ is the matrix that reflects across the line $\mathbf{v}$ and $2(I - P\mathbf{v}) - I = I - 2P_{\mathbf{v}} = H_{\mathbf{v}}$ is the matrix that reflects across the hyperplane $\mathbf{v}^{\perp}$. Since $P_{\mathbf{v}}^2 = P_{\mathbf{v}}$ and $P_{\mathbf{v}}^T = P_{\mathbf{v}}$, we find that

$$H_{\mathbf{v}}^{-1} = H_{\mathbf{v}} \quad \text{and } H_{\mathbf{v}}^T = H_{\mathbf{v}}.$$

In particular, $H_{\mathbf{v}}$ is an orthogonal matrix.

The key trick of the Householder algorithm is that $H_{\mathbf{v}}\mathbf{a} = \mathbf{r}$ for any $\mathbf{a}, \mathbf{r}$ satisfying $\|\mathbf{a}\| = \|\mathbf{r}\|$ and $\mathbf{a} - \mathbf{r} = \mathbf{v}$. Picture:



Here is the algorithm.

Householder QR. We are given an invertible matrix A with first column $\mathbf{a} \in \mathbb{R}^n$.

- Let $\mathbf{r} := (\|\mathbf{a}\|, 0, \ldots, 0)$, $\mathbf{v} := \mathbf{a} - \mathbf{r}$ and $H_1 := H_{\mathbf{v}}$, so that $H_1\mathbf{a} = \mathbf{r}$. Then we have

$$H_1 A = \left(\begin{array}{c|c|c|c} H_1\mathbf{a} & * & \cdots & * \end{array} \right) = \left(\begin{array}{c|c|c|c} \mathbf{r} & * & \cdots & * \end{array} \right) = \left(\begin{array}{c|ccc} \|\mathbf{a}\| & * & \cdots & * \\ \hline 0 & & & \\ \vdots & & A' & \\ 0 & & & \end{array} \right),$$

for some matrix A' of size $(n-1) \times (n-1)$.

- Let $\mathbf{a}' \in \mathbb{R}^{n-1}$ be the first column of A', let $\mathbf{r}' = (\|\mathbf{a}'\|, 0, \ldots, 0) \in \mathbb{R}^{n-1}$ and let $\mathbf{v}' = \mathbf{a}' - \mathbf{r}'$, so that $H_{\mathbf{v}'}\mathbf{a}' = \mathbf{r}'$. Then the matrix

$$H_2 := \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & H_{\mathbf{v}'} & \\ 0 & & & \end{array}\right),$$

satisfies

$$\begin{aligned} H_2 H_1 A &= \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & H_{\mathbf{v}'} & \\ 0 & & & \end{array}\right) \left(\begin{array}{c|ccc} \|\mathbf{a}\| & * & \cdots & * \\ \hline 0 & & & \\ \vdots & & A' & \\ 0 & & & \end{array}\right) \\ &= \left(\begin{array}{c|ccc} \|\mathbf{a}\| & * & \cdots & * \\ \hline 0 & & & \\ \vdots & & H_2 A' & \\ 0 & & & \end{array}\right) \\ &= \left(\begin{array}{c|c|ccc} \|\mathbf{a}\| & * & * & \cdots & * \\ \hline 0 & \|\mathbf{a}'\| & * & \cdots & * \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & A'' & \\ 0 & 0 & & & \end{array}\right), \end{aligned}$$

for some matrix A'' of size $(n-2) \times (n-2)$. We observe that the matrix H_2 is itself a Householder reflection matrix. To see this, let $\mathbf{w} = \left(\begin{array}{c|c} 0 & (\mathbf{v}')^T \end{array}\right)^T$, so that $\|\mathbf{w}\| = \|\mathbf{v}'\|$. Then we have

$$\begin{aligned} H_{\mathbf{w}} &= I - 2\frac{\mathbf{w}\mathbf{w}^T}{\|\mathbf{w}\|} \\ &= I - \frac{2}{\|\mathbf{v}'\|} \left(\begin{array}{c} 0 \\ \hline \mathbf{v}' \end{array}\right) \left(\begin{array}{c|c} 0 & (\mathbf{v}')^T \end{array}\right) \\ &= \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & I_{n-1} & \\ 0 & & & \end{array}\right) - \left(\begin{array}{c|ccc} 0 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & -2\frac{\mathbf{v}'(\mathbf{v}')^T}{\|\mathbf{v}'\|} & \\ 0 & & & \end{array}\right) \end{aligned}$$

$$
\begin{aligned}
&= \left(\begin{array}{c|c} 1 & 0 \quad \cdots \quad 0 \\ \hline 0 & \\ \vdots & I_{n-1} - 2\frac{\mathbf{v}'(\mathbf{v}')^T}{\|\mathbf{v}'\|} \\ 0 & \end{array}\right) \\
&= \left(\begin{array}{c|c} 1 & 0 \quad \cdots \quad 0 \\ \hline 0 & \\ \vdots & H_{\mathbf{v}'} \\ 0 & \end{array}\right) \\
&= H_2.
\end{aligned}
$$

- Continuing in this way for $n-1$ steps gives an upper triangular matrix:

$$
H_{n-1}\cdots H_2 H_1 A = \begin{pmatrix} \|\mathbf{a}\| & * & \cdots & \cdots & * \\ & \|\mathbf{a}'\| & & & \vdots \\ & & \ddots & & \vdots \\ & & & \|\mathbf{a}^{(n-1)}\| & * \\ & & & & b \end{pmatrix} = R,
$$

  where each H_i is a Householder matrix $H_{\mathbf{v}_i}$ for some vector $\mathbf{v}_i \in \mathbb{R}$. Note that the diagonal entries of R are nonzero since we have assumed that A is invertible. The real number b can be positive or negative.

- Finally, since each Householder reflection is equal to its own inverse, we obtain

$$
\begin{aligned}
H_{n-1}\cdots H_2 H_1 A &= R \\
A &= H_1 H_2 \cdots H_{n-1} R \\
A &= QR.
\end{aligned}
$$

As a consequence, we will prove that every real orthogonal matrix is a composition of reflections. Suppose that $A^T A = I$ and consider the Householder factorization

$$
H_{n-1}\cdots H_2 H_1 A = R.
$$

Now each matrix on the left is orthogonal. Since a product of orthogonal matrices is orthogonal, we conclude that R is also orthogonal. In particular, the rows of R are orthonormal. Since R is also upper-triangular, this implies that R is diagonal:

$$
R = \begin{pmatrix} \|\mathbf{a}\| & & & & \\ & \|\mathbf{a}'\| & & & \\ & & \ddots & & \\ & & & \|\mathbf{a}^{(n-1)}\| & \\ & & & & b \end{pmatrix}
$$

Finally, since each row of R has length 1, we conclude that

$$R = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & \pm 1 \end{pmatrix}$$

If the last entry is $+1$ then $R = I$ and we obtain

$$A = H_1 H_2 \cdots H_{n-1} I = H_1 H_2 \cdots H_{n-1},$$

which shows that A is a product of $n-1$ reflection matrices. If the last entry of R is -1, then $H_n := R$ equals the Householder matrix $H_{\mathbf{e}_n}$, where $\mathbf{e}_n = (0, \ldots, 0, 1)$. In this case we see that A is a product of n reflection matrices:

$$A = H_1 H_2 \cdots H_{n-1} R = H_1 H_2 \cdots H_{n-1} H_n.$$

Remark: There is some restriction on n. Since each reflection matrix has determinant -1, a product of n reflection matrices has determinant $(-1)^n$. Hence an orthogonal matrix A satisfying $\det(A) = +1$ can only be expressed as an even product of reflections and an orthogonal matrix satisfying $\det(A) = -1$ can only be expressed as an odd product of reflections.

2.2 Schur Triangularization

Given a square matrix A, we always want to find a simpler matrix B that is *similar* to A. That is, we want to find a simpler matrix B and an invertible matrix X such that $A = XBX^{-1}$. Then for any polynomial function $f(x)$ (more generally, for power series or Laurent polynomials) we can compute

$$f(A) = X \cdot f(B) \cdot X^{-1}.$$

The nicest possible situation is when B is diagonal and X is orthogonal or unitary: $X^{-1} = X^T$ or $X^{-1} = X^*$. This is the subject of the Spectral Theorem in the next section. But diagonalization is not always possible. There are three different theorems for dealing with non-diagonalizable matrices:

- Schur triangularization.
- Jordan normal form.
- Singular value decomposition.

We will deal with all three of these in this chapter. We begin with Schur triangularization.

We say that a matrix is upper-triangular if all entries below the main diagonal are zero:[31]

$$T = \begin{pmatrix} t_{11} & * & \cdots & * \\ & t_{22} & & \vdots \\ & & \ddots & * \\ & & & t_{nn} \end{pmatrix}$$

These matrices have some nice properties:

- The eigenvalues of T are the diagonal entries. Indeed, the characteristic polynomial is

$$\chi_T(x) = (x - t_{11})(x - t_{22}) \cdots (x - t_{nn}).$$

- Products and sums of upper-triangular matrices behave as products and sums for the diagonal entries. Thus for any polynomial $f(x)$ we have

$$f(T) = \begin{pmatrix} f(t_{11}) & * & \cdots & * \\ & f(t_{22}) & & \vdots \\ & & \ddots & * \\ & & & f(t_{nn}) \end{pmatrix}$$

  Unfortunately, the entries above the diagonal are messy.

- If T is invertible then T^{-1} is also upper-triangular, and the previous formula also applies for Laurent polynomials $f(x)$.

- If the largest eigenvalue satisfies $|\lambda| < 1$ then one can show that $T^k \to O$ as $k \to \infty$, though the proof is a bit tricky.[32]

Here is our main theorem.

Theorem (Schur Triangularization). For **any** square matrix A over $\mathbb{R}$ or $\mathbb{C}$, there exists an upper-triangular matrix T and a unitary matrix $U^{-1} = U^*$ such that

$$\begin{aligned} A &= UTU^{-1} \\ A &= UTU^* \end{aligned}$$

$$\left(\begin{array}{c|c|c} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{array} \right) = \left(\begin{array}{c|c|c} \mathbf{u}_1 & \ldots & \mathbf{u}_n \end{array} \right) \begin{pmatrix} t_{11} & * & \cdots & * \\ & t_{22} & & \vdots \\ & & \ddots & * \\ & & & t_{nn} \end{pmatrix} \begin{pmatrix} \mathbf{u}_1^* \\ \hline \vdots \\ \hline \mathbf{u}_n^* \end{pmatrix}.$$

Even if A is real, the matrices U and T will generally have complex entries. However, if A is a real matrix with real eigenvalues then we can choose U and T to be real.

[31] Similarly, a lower-triangular matrix has zeros above the main diagonal.
[32] This is easy for diagonalizable matrices.

Proof. We use induction on the size of A. First we note that the theorem is trivially true for 1×1 matrices, i.e., for scalars: : $(a) = (1)(a)(1)$. Now let A have shape $n \times n$ for some $n \geq 2$. We have seen that every real matrix has a (possibly complex) eigenvalue. Let $t_{11} \in \mathbb{C}$ be an eigenvalue of A and let $\mathbf{u}_1$ be a corresponding eigenvector of length 1.[33] Now let U_1 be any unitary matrix with first column $\mathbf{u}_1$:

$$U_1 = \left(\begin{array}{c|c|c} \mathbf{u}_1 & \cdots & \mathbf{u}_n \end{array} \right).$$

To find such a matrix, we first complete $\mathbf{u}_1$ to a basis, $\mathbf{u}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$, then apply Gram-Schmidt to convert this into an orthonormal basis $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$.[34] Since these vectors satisfy $\mathbf{u}_1^* \mathbf{u}_1 = 1$ and $\mathbf{u}_i^* \mathbf{u}_1 = 0$ for $i \geq 2$, we observe that

$$\begin{aligned}
U_1^* A U_1 &= \begin{pmatrix} \mathbf{u}_1^* \\ \hline \vdots \\ \hline \mathbf{u}_n^* \end{pmatrix} \left(\begin{array}{c|c|c|c} A\mathbf{u}_1 & A\mathbf{u}_2 & \cdots & A\mathbf{u}_n \end{array} \right) \\
&= \begin{pmatrix} \mathbf{u}_1^* \\ \hline \vdots \\ \hline \mathbf{u}_n^* \end{pmatrix} \left(\begin{array}{c|c|c|c} t_{11}\mathbf{u}_1 & A\mathbf{u}_2 & \cdots & A\mathbf{u}_n \end{array} \right) \\
&= \left(\begin{array}{c|c|c|c} \begin{matrix} t_{11}\mathbf{u}_1^*\mathbf{u}_1 \\ t_{11}\mathbf{u}_2^*\mathbf{u}_1 \\ \vdots \\ t_{11}\mathbf{u}_n^*\mathbf{u}_1 \end{matrix} & * & \cdots & * \end{array} \right) \\
&= \left(\begin{array}{c|ccc} t_{11} & * & \cdots & * \\ \hline 0 & & & \\ \vdots & & A_2 & \\ 0 & & & \end{array} \right),
\end{aligned}$$

for some matrix A_2 of shape $(n-1) \times (n-1)$. By induction there exists an $(n-1) \times (n-1)$ unitary matrix U_2 such that $T_2 := U_2^* A_2 U_2$ is upper-triangular. Now define the matrix

$$U := U_1 \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & U_2 & \\ 0 & & & \end{array} \right).$$

[33] Just take any eigenvector and scale it. If t_{11} is real and if A has real entries then we can choose $\mathbf{u}_1$ to have real entries, in which case we can choose U_1 to have real entries.

[34] Any set of independent vectors can be completed to a basis using Steinitz exchange.

We observe that this matrix is unitary:

$$
\begin{aligned}
U^*U &= \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & U_2^* & \\ 0 & & & \end{array}\right) U_1^* U_1 \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & U_2 & \\ 0 & & & \end{array}\right) \\
&= \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & U_2^* & \\ 0 & & & \end{array}\right) \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & U_2 & \\ 0 & & & \end{array}\right) \\
&= \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & U_2^* U_2 & \\ 0 & & & \end{array}\right) \\
&= \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & I_{n-1} & \\ 0 & & & \end{array}\right) \\
&= I_n.
\end{aligned}
$$

And we observe that the matrix $T := U^*AU$ is upper triangular, as desired:

$$
\begin{aligned}
T &= U^*AU \\
&= \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & U_2^* & \\ 0 & & & \end{array}\right) U_1^* A U_1 \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & U_2 & \\ 0 & & & \end{array}\right) \\
&= \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & U_2^* & \\ 0 & & & \end{array}\right) \left(\begin{array}{c|ccc} t_{11} & * & \cdots & * \\ \hline 0 & & & \\ \vdots & & A_2 & \\ 0 & & & \end{array}\right) \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & U_2 & \\ 0 & & & \end{array}\right) \\
&= \left(\begin{array}{c|ccc} t_{11} & * & \cdots & * \\ \hline 0 & & & \\ \vdots & & U_2^* A_2 U_2 & \\ 0 & & & \end{array}\right)
\end{aligned}
$$

$$= \left(\begin{array}{c|ccc} t_{11} & * & \cdots & * \\ \hline 0 & & & \\ \vdots & & T_2 & \\ 0 & & & \end{array}\right).$$

□

Before moving on, I will mention one application. If $A = XTX^{-1}$ for some (upper or lower) triangular matrix T, then the eigenvalues of A are the diagonal entries of T. One could imagine using this to **compute** the eigenvalues of A. Unfortunately, the proof of Schur triangularization assumes that we already know the eigenvalues of A.

Nevertheless, this is still the good idea, and it is behind the most powerful algorithm for computing eigenvalues. This algorithm uses the QR factorization (which does not assume knowledge of the eigenvalues) in a surprising way to recursively approximate the Schur decomposition, and hence the eigenvalues. It was discovered in the late 1950s by Francis and Kublanovskaya. I will present only the most basic version. The real world version uses extra tricks and optimizations.

The QR Algorithm for Computing Eigenvalues. Given a square matrix A, we recursively define unitary matrices $Q_1, Q_2, \ldots$ and upper-triangular matrices $R_1, R_2, \ldots$ as follows:

- Compute a QR factorization: $A = Q_1R_1$.
- Next, compute a QR factorization of the matrix R_1Q_1:[35]

$$R_1Q_1 = Q_2R_2.$$

- Continue to compute Q_{k+1} and R_{k+1} from the matrix R_kQ_k:

$$R_kQ_k = Q_{k+1}R_{k+1}.$$

Let's write $A_1 := A = Q_1R_1$ and $A_k := Q_kR_k$. Since the Q in the QR factorization is unitary, we have $R_k = Q_k^*A_k$ and hence

$$A_{k+1} = R_kQ_k = Q_k^*A_kQ_k.$$

This implies that the sequence of matrices $A = A_1, A_2, \ldots$ all have the same eigenvalues. The theorem says the following.

Theorem. Suppose that A has eigenvalues with distinct absolute values:[36]

$$|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|.$$

[35] This is a strange idea, but it leads to great results.

[36] There are modified versions of the algorithm that work for all square matrices.

Then the matrix $A_k = R_k Q_k$ in the QR algorithm converges to an upper triangular matrix, whose diagonal entries are the eigenvalues of A.

It is difficult to find a proof of this written down.[37] The only full proof I can find is in Wilkinson, *The Algebraic Eigenvalue Problem* (1965), page 516. Here is a sketch.

Sketch of a Proof. Define $\tilde{Q}_k := Q_1 Q_2 \cdots Q_k$ and $\tilde{R}_k = R_1 R_2 \cdots R_k$. Since the sets of unitary matrices and upper triangular matrices are closed under multiplication,[38] we see that $\tilde{Q}_k$ is unitary and $\tilde{R}_k$ is upper triangular. I claim that $A_{k+1} = \tilde{Q}_k^* A \tilde{Q}_k$ and $A^k = \tilde{Q}_k \tilde{R}_k$. Indeed, we have

$$\begin{aligned}
A_{k+1} &= Q_k^* A_k Q_k \\
&= Q_k^* Q_{k-1}^* A_{k-1} Q_{k-1} Q_k \\
&\ \vdots \\
&= Q_k^* \cdots Q_2^* Q_1^* A Q_1 Q_2 \cdots Q_k \\
&= (Q_1 \cdots Q_k)^* A (Q_1 \cdots Q_k) \\
&= \tilde{Q}_k^* A \tilde{Q}_k
\end{aligned}$$

and

$$\begin{aligned}
A^k &= (Q_1 R_1) \cdots (Q_1 R_1) \\
&= Q_1 (R_1 Q_1) \cdots (R_1 Q_1) R_1 \\
&= Q_1 (Q_2 R_2) \cdots (Q_2 R_2) R_1 \\
&= Q_1 Q_2 (R_2 Q_2) \cdots (R_2 Q_2) R_2 R_1 \\
&= Q_1 Q_2 (Q_3 R_3) \cdots (Q_3 R_3) R_2 R_1 \\
&\ \vdots \\
&= (Q_1 Q_2 \cdots Q_k)(R_k \cdots R_2 R_1) \\
&= \tilde{Q}_k \tilde{R}_k.
\end{aligned}$$

From our assumption that A has distinct eigenvalues, we can diagonalize A as

$$A = X \Lambda X^{-1} = X \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} X^{-1}.$$

By multiplying X^{-1} on the left by elementary matrices we can write $X^{-1} = L\Gamma$ where Γ is upper triangular and L is lower triangular with 1s on the diagonal. The key to the whole proof is to observe that $\Lambda^k L \Lambda^{-k}$ is lower triangular and converges to the identity matrix as

[37] Pure math books tend not to discuss it and applied math books tend not to prove it.

[38] Jargon: These sets are groups.

$k \to \infty$. Indeed, if ℓ_{ij} is the ij entry of L (so that $\ell_{ii} = 1$ and $\ell_{ij} = 0$ when $i < j$) then the ij entry of $\Lambda^k L \Lambda^{-k}$ is

$$(\Lambda^k L \Lambda^{-k})_{ij} = \begin{cases} 0 & i < j, \\ 1 & i = j, \\ \ell_{ij}(\lambda_i/\lambda_j)^k & i > j. \end{cases}$$

Since we have assumed that $|\lambda_1| > \cdots > |\lambda_n|$, it follows that the entries below the diagonal go to zero as $k \to \infty$.

By combining these ingredients, Wilkinson shows that the sequences $\tilde{Q}_k$ and $\tilde{R}_k$ converge, and that the the sequence Q_k converges to a diagonal matrix, hence $A_k = Q_k R_k$ converges to an upper triangular matrix. Let's say $\tilde{Q}_k \to U$ and $A_k \to T$, for unitary U and upper triangular T. Then in the limit we obtain the Schur triangularization:

$$A = \tilde{Q}_k A_{k+1} \tilde{Q}_k^* \to UTU^*.$$

□

Remark: The proof uses the fact QR factorization is unique up to multiplication with a unitary diagonal matrix D: $QR = (QD)(D^{-1}R)$. This follows from the the fact that any unitary upper triangular matrix must be diagonal.

2.3 The Spectral Theorem

Write a section on normal matrices in the previous chapter

The spectral theorem deals with the best kinds of matrices.

We've looked at polynomials and power series. What about transpose and conjugate transpose?

Normal matrices. (Maybe do this in a separate section?)

2.4 The Singular Value Decomposition

A form of "generalized diagonalization" that applies to rectangular matrices and non-diagonalizable square matrices.

For any $m \times n$ matrix A and $n \times m$ matrix B, the square matrices AB ($m \times m$) and BA ($n \times n$) have the same nonzero eigenvalues. If $m < n$ then the matrix BA has $n - m$ extra zero eigenvalues compared to AB.

For any $m \times n$ matrix A, the eigenvalues of $A^T A$ are real and non-negative: $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$. The singular values of A are the non-negative real square roots of the eigenvalues: $\sigma_i = \sqrt{\lambda_i}$. Equivalently, $\sigma_1^2, \ldots, \sigma_n^2$ are the eigenvalues of $A^T A$.

Properties: The largest singular value σ_1 is the operator norm $\|A\|$. The product of the singular values is $\sqrt{\det(A^T A)}$.

Let Σ be the $n \times n$ diagonal matrix of singular values, so $\Lambda = \Sigma^2 = \Sigma^T \Sigma = \Sigma \Sigma^T$ is the diagonal matrix of eigenvalues. From the spectral theorem there exists a unitary (orthogonal) matrix V such that $A^T A = V \Lambda V^T = V \Sigma \Sigma^T V^T = (V\Sigma)(V\Sigma)^T$. The columns $\mathbf{v}_i$ of V are the eigenvectors of $A^T A$.

Suppose that $A^T A$ has rank r, which is also the rank of A and AA^T, so that there are r nonzero singular values $\sigma_1 \geq \cdots \geq \sigma_r$. From our first remark, $A^T A$ and AA^T have the same non-zero eigenvalues. Define $\mathbf{u}_i = (A\mathbf{v}_i)/\sigma_i$ for $1 \leq i \leq r$. Then $\mathbf{u}_i$ are the eigenvectors of AA^T corresponding to the eigenvalues $\sigma_1^2 \geq \cdots \geq \sigma_r^2$. Complete the $\mathbf{u}_i$ to a basis of $\mathbb{R}^m$ arbitrarily and let U the $m \times m$ unitary (orthogonal) matrix with columns $\mathbf{u}_i$. Then we have $A = U\Sigma V^*$. (I guess we have to pad Σ with some zeros.) This is the singular value decomposition (SVD).

Geometry: A sends the unit ball in $\mathbb{R}^n$ to an ellipsoid in $\mathbb{R}^m$. The singular values of A are the radii of the ellipsoid.

Eckart-Young Theorem. Write $A = \sum_{i=1}^r \sigma_i \mathbf{u}_i \mathbf{v}_i^T$. Then $A_k = \sum_{i=1}^k \sigma_i \mathbf{u}_i \mathbf{v}_i^T$ is the best rank k approximation to A. That is, for any rank k matrix B we have $\|A - A_k\| \leq \|A - B\|$.

Application: Principal Component Analysis. (Total Least Squares.)

Maybe put all of this in a separate chapter.

2.5 Jordan Canonical Form

The companion matrix.

3 Applications of Spectral Theory

3.1 The Principal Axes Theorem

The earliest example of the Spectral Theorem goes all the way back to the birth of analytic geometry. It was known to Descartes and Fermat in the early 1600s and was applied by Euler in the 1700s to the mechanics of rotating bodies.[39]

You may have seen this theorem in school: Any polynomial equation of the form

$$f(x, y) = a + bx + cy + dx^2 + exy + fy^2 = 0$$

can be brought into standard form by a translation and a rotation. The standard forms are

$$\begin{aligned} \text{parabola} &: y = ax^2 \text{ or } x = ay^2, \\ \text{ellipse} &: x^2/a^2 + y^2/b^2 = 1, \\ \text{hyperbola} &: \pm(x^2/a^2 - y^2/b^2) = 1. \end{aligned}$$

The Principal Axes Theorem generalizes this to higher dimensions.

Theorem (Principal Axes Theorem). Consider a general polynomial $f(x_1, \ldots, x_n)$ of degree 2 in n variables. This can be expressed as

$$f(\mathbf{x}) = b + \mathbf{b}^T \mathbf{x} + \mathbf{x}^T B \mathbf{x},$$

[39] *Highlights in the History of Spectral Theory*, Steen.

for some scalar b, vector $\mathbf{b}$ and **symmetric** matrix B. If B^{-1} exists, then we can find a change of variables $\mathbf{u} = Q\mathbf{x} + \mathbf{t}$, where $Q^TQ = I$ is an orthogonal matrix[40] and $\mathbf{t}$ is a (translation) vector, such that

$$f(u_1, \dots, u_n) = a + \lambda_1 u_1^2 + \lambda_2 u^2 + \cdots + \lambda_n u_n^2.$$

What if B^{-1} doesn't exist?

Proof. Let $\mathbf{u} = Q\mathbf{x} + \mathbf{t}$ for some invertible matrix Q and vector $\mathbf{t}$. Then we have

$$\begin{aligned}
f(\mathbf{u}) &= f(Q\mathbf{x} + \mathbf{t}) \\
&= b + \mathbf{b}^T(Q\mathbf{x} + \mathbf{t}) + (Q\mathbf{x} + \mathbf{t})^T B(Q\mathbf{x} + \mathbf{t}) \\
&= b + \mathbf{b}^T Q\mathbf{x} + \mathbf{b}^T\mathbf{t} + (\mathbf{x}^T Q^T + \mathbf{t}^T)B(Q\mathbf{x} + \mathbf{t}) \\
&= b + \mathbf{b}^T Q\mathbf{x} + \mathbf{b}^T\mathbf{t} + \mathbf{x}^T Q^T BQ\mathbf{x} + \mathbf{x}^T Q^T B\mathbf{t} + \mathbf{t}^T BQ\mathbf{x} + \mathbf{t}^T B\mathbf{t} \\
&= b + \mathbf{b}^T Q\mathbf{x} + \mathbf{b}^T\mathbf{t} + \mathbf{x}^T Q^T BQ\mathbf{x} + 2\mathbf{t}^T BQ\mathbf{x} + \mathbf{t}^T B\mathbf{t} \qquad (*) \\
&= (b + \mathbf{b}^T\mathbf{t} + \mathbf{t}^T B\mathbf{t}) + (\mathbf{b}^T Q + 2\mathbf{t}BQ)\mathbf{x} + \mathbf{x}^T Q^T BQ\mathbf{x}.
\end{aligned}$$

Step $(*)$ uses the facts that $B^T = B$ and that $\mathbf{t}^T BQ\mathbf{x}$ is a scalar, hence

$$t^T BQ\mathbf{x} = (t^T BQ\mathbf{x})^T = \mathbf{x}^T Q^T B^T\mathbf{t} = \mathbf{x}^T Q^T B\mathbf{x}.$$

If B^{-1} exists, then we can eliminate the linear terms by taking

$$\begin{aligned}
\mathbf{b}^T Q + 2\mathbf{t}BQ &= \mathbf{0}^T \\
2\mathbf{t}BQ &= -\mathbf{b}^T Q \\
\mathbf{t}BQ &= -\frac{1}{2}\mathbf{b}^T Q \\
\mathbf{t} &= -\frac{1}{2}\mathbf{b}^T QQ^{-1}B^{-1} \\
\mathbf{t} &= -\frac{1}{2}\mathbf{b}^T B^{-1}.
\end{aligned}$$

Finally, since B is symmetric, the Spectral Theorem says that we can choose orthogonal $Q^TQ = I$ so that $Q^T BQ$ is diagonal:

Choose $\mathbf{t}$ so that $\mathbf{b}^T + 2\mathbf{t}^T B = \mathbf{0}^T$. Assume B invertible.

3.2 Positive Definite Matrices

If $\langle \mathbf{x}, B\mathbf{x}\rangle \geq 0$ (enough to assume $\in \mathbb{R}$) for all $\mathbf{x} \in \mathbb{C}^n$ then $B^* = B$.

Proof: We need to show that $\langle B\mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{x}, B\mathbf{y}\rangle$ for all $\mathbf{x}, \mathbf{y}$. First note that $\langle B\mathbf{x}, \mathbf{x}\rangle = \langle \mathbf{x}, B\mathbf{x}\rangle^* = \langle \mathbf{x}, B\mathbf{x}\rangle$, so $\langle \mathbf{x}, (B - B^*)\mathbf{x}\rangle$ for all $\mathbf{x}$. We need to show that $\langle \mathbf{x}, T\mathbf{x}\rangle = 0$ for all $\mathbf{x}$ implies $T = O$. Indeed, we have

$$0 = \langle \mathbf{x} + \mathbf{y}, T(\mathbf{x} + \mathbf{y})\rangle = \langle \mathbf{x}, T\mathbf{y}\rangle + \langle \mathbf{y}, T\mathbf{x}\rangle + 0 + 0$$

[40] If the coefficients of f are complex then $Q^*Q = I$ is unitary, but we are usually interested in the real case.

and

$$0 = \langle \mathbf{x} + i\mathbf{y}, T(\mathbf{x} + i\mathbf{y}) \rangle = i\langle \mathbf{x}, T\mathbf{y} \rangle - i\langle \mathbf{y}, T\mathbf{x} \rangle + 0 + 0.$$

Divide the second equation by i and add them to obtain $2\langle \mathbf{x}, T\mathbf{y} \rangle = 0$ and hence $\langle \mathbf{x}, T\mathbf{y} \rangle = 0$ for all $\mathbf{x}, \mathbf{y}$.

In principle, our proof of the Spectral Theorem gives an algorithm to factor a semi-definite matrix $B = A^T A$, but is probably not the most efficient method since it assumes that we already know the eigenvalues. The *Cholesky factorization* is a method to factor $B = A^T A$ that avoids having to compute eigenvalues.

3.3 Differential Equations

The matrix exponential encodes the solution to linear systems of differential equations. To begin, recall the power series definition of the exponential function:

$$\exp(x) := 1 + x + \frac{1}{2}x^2 + \cdots + \frac{1}{k!}x^k + \cdots .$$

It is a basic theorem of analysis that this series converges uniformly for any complex number $x \in \mathbb{C}$. It was invented by Euler because of the following special properties. For any complex numbers $x, y \in \mathbb{C}$ we have

$$\begin{aligned}
\exp(x)\exp(y) &= \left(\sum_{i \geq 0} \frac{1}{i!} \cdot x^i \right) \left(\sum_{j \geq 0} \frac{1}{j!} \cdot x^j \right) \\
&= \sum_{k \geq 0} \left(\sum_{i+j=k} \frac{1}{i!} \cdot x^i \cdot \frac{1}{j!} \cdot x^j \right) \\
&= \sum_{k \geq 0} \frac{1}{k!} \cdot \left(\sum_{i+j=k} \frac{k!}{i!j!} x^i x^j \right) \\
&= \sum_{k \geq 0} \frac{1}{k!} \cdot (x+y)^k \\
&= \exp(x+y).
\end{aligned}$$

This property suggests that $\exp(x) = e^x$ for some number e, which Euler calculated to be ≈ 2.71828. Furthermore, the power series $\exp(x)$ is equal to its own derivative:

$$\begin{aligned}
\frac{d}{dx}\exp(x) &= \frac{d}{dx}\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots \right) \\
&= 0 + 1 + \frac{1}{2!} \cdot 2x + \frac{1}{3!} \cdot 3x^2 + \frac{1}{4!} \cdot 4x^3 + \cdots \\
&= 0 + 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots \\
&= \exp(x).
\end{aligned}$$

Conversely, let $f : \mathbb{C} \to \mathbb{C}$ be any function satisfying $\frac{d}{dx} f(x) = f(x)$. Suppose that $f(x)$ has a convergent power series expansion near $x = 0$:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots .$$

The equation $\frac{d}{dx} f(x) = f(x)$ tells us that

$$a_0 + a_1 x + a_2 x^2 + \cdots = a_1 + 2a_1 x + 3a_2 x^2 + \cdots .$$

Then comparing coefficients tells us that $a_k = (k+1)a_{k-1}$ for all $k \geq 0$, which has the unique solution $a_k = a_0/k!$. Hence we must have $f(x) = a_0 \exp(x)$.

Now consider a vector of functions $\mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))$. We can think of this as a parametrized path in n-dimensional space: $\mathbf{x} : \mathbb{R} \to \mathbb{R}^n$.[41] A linear system of ordinary differential equations has the form

$$\left\{ \begin{array}{rcccccc} x_1'(t) & = & a_{11}x_1(t) & + & \cdots & + & a_{1n}x_n(t) \\ & \vdots & & & & & \\ x_n'(t) & = & a_{n1}x_1(t) & + & \cdots & + & a_{nn}x_n(t) \end{array} \right\} \quad \rightsquigarrow \quad \mathbf{x}'(t) = A\mathbf{x}(t),$$

for some $n \times n$ matrix A of constants. We can think of $\mathbf{x}'(t)$ as the velocity vector of the path $\mathbf{x}(t)$, and we can think of A as specifying a vector field on $\mathbb{R}^n$, with value $A\mathbf{x}$ at the point $\mathbf{x}$. A solution to the equation $\mathbf{x}'(t) = A\mathbf{x}(t)$ is any path $\mathbf{x}(t)$ in $\mathbb{R}^n$ that flows along the vector field defined by A. For any initial point $\mathbf{x}(0) \in \mathbb{R}^n$

The companion matrix:

`https://math.stackexchange.com/questions/348498/jordan-basis-of-a-when-a-is-the-companion`

3.4 Markov Chains

Perron-Frobenius, Page Rank

3.5 Singular Value Decomposition

3.6 Total Least Squares

Given a matrix of n data points in any dimensional space:

$$X = \left(\begin{array}{c|c|c} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{array} \right)$$

For any vector $\mathbf{a}$ let $P_{\mathbf{a}} = \mathbf{a}\mathbf{a}^T/\|\mathbf{a}\|^2$ be projection onto the line $\mathbf{a}$ and $Q_{\mathbf{a}} = I - P_{\mathbf{a}}$ be projection onto the hyperplane $\mathbf{a}^\perp$. For any i we have

$$\mathbf{x}_i = P_{\mathbf{a}}\mathbf{x}_i + Q_{\mathbf{a}}\mathbf{x}_i$$
$$\|\mathbf{x}_i\|^2 = \|P_{\mathbf{a}}\mathbf{x}_i\|^2 + \|Q_{\mathbf{a}}\mathbf{x}_i\|^2$$

[41] I guess we'll work with real numbers to make visualization easier.

$$\sum_i \|\mathbf{x}_i\|^2 = \sum \|P_{\mathbf{a}}\mathbf{x}_i\|^2 + \sum \|Q_{\mathbf{a}}\mathbf{x}_i\|^2.$$

Goal: Choose $\mathbf{a}$ to minimize $\sum \|Q_{\mathbf{a}}\mathbf{x}_i\|^2$. Since $\sum_i \|\mathbf{x}_i\|^2$ is fixed by the data, this is the same as maximizing $\sum \|P_{\mathbf{a}}\mathbf{x}_i\|^2$. But

$$\|P_{\mathbf{a}}\mathbf{x}_i\|^2 = \frac{1}{\|a\|^2}|\mathbf{a}^T\mathbf{x}_i|^2 = \frac{1}{\|\mathbf{a}\|^2}\mathbf{a}^T\mathbf{x}_i\overline{\mathbf{a}^T\mathbf{x}_i} = \frac{1}{\|\mathbf{a}\|^2}\mathbf{a}^T\mathbf{x}_i\mathbf{x}_i^T\mathbf{a},$$

hence

$$\sum \|P_{\mathbf{a}}\mathbf{x}_i\|^2 = \frac{1}{\|\mathbf{a}\|^2}\mathbf{a}^T XX^T\mathbf{a} = \frac{1}{\|\mathbf{a}\|^2}(X^T\mathbf{a})^T(X^T\mathbf{a}) = \frac{\|X^T\mathbf{a}\|^2}{\|\mathbf{a}\|^2}.$$

This is maximized by letting $\mathbf{a}$ be an eigenvector for the largest (real) eigenvalue of XX^T.

Proof. By S.T., XX^T can be unitarily diagonalized: $XX^T\mathbf{u}_i = \sigma_i^2\mathbf{u}_i$. Let

$$\mathbf{a} = c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n,$$

so that

$$\frac{1}{\|\mathbf{a}\|^2}\mathbf{a}^T XX^T\mathbf{a} = \frac{\sigma_1^2 c_1^2 + \cdots + \sigma_n^2 c_n^2}{c_1^2 + \cdots + c_n^2}.$$

Maximum when $c_1 = 1$ and $c_2 = \cdots = c_n = 0$. Maximum under constraint $c_1 = 0$ gives $c_2 = 1$ and $c_3 = \cdots = c_n = 0$, etc.