Math 510 **Fall 2022**
Homework 5 **Drew Armstrong**

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1. Trace and Determinant. The characteristic polynomial of a square matrix satisfies

$$\chi_A(x) = \det(xI - A) = x^n - \mathrm{tr}(A)x^{n-1} + \cdots + (-1)^n \det(A),$$

where $\det(A)$ is the determinant and $\mathrm{tr}(A)$ is the trace, i.e., the sum of the diagonal entries.

(a) If $A = XBX^{-1}$ for some matrices B and X, prove that $\chi_A(x) = \chi_B(x)$. Use this to show that $\det(A) = \det(B)$ and $\mathrm{tr}(A) = \mathrm{tr}(B)$.

(b) From the Fundamental Theorem of Algebra we know that $\chi_A(x)$ factors as

$$\chi_A(x) = (x - \lambda_1)(x - \lambda_2)\cdots(x - \lambda_n)$$

for some complex numbers $\lambda_1, \ldots, \lambda_n$, not necessarily distinct. In this case, show that

$$\mathrm{tr}(A) = \lambda_1 + \cdots + \lambda_n \quad \text{and} \quad \det(A) = \lambda_1 \cdots \lambda_n.$$

2. Non-Real Eigenvalues of a Real Matrix. Let A be a real $n \times n$ matrix with real entries and consider the characteristic polynomial

$$f(x) = \det(xI_n - A).$$

(a) For any complex number $\alpha \in \mathbb{C}$ show that $f(\alpha)^* = f(\alpha^*)$. [Hint: Actually, this holds for any polynomial $f(x)$ with real coefficients. Use properties of conjugation.]

(b) Use (a) to show that $f(\alpha) = 0$ if and only if $f(\alpha^*) = 0$.

(c) Use (b) to show that the non-real eigenvalues of A come in pairs.

(d) If n is odd, use (c) to show that A must have a real eigenvalue.

3. Idempotent Matrices. Let P be an $n \times n$ matrix satisfying $P^2 = P$.

(a) Show that the eigenvalues of P are in the set $\{0, 1\}$.

(b) You may assume without proof that P is diagonalizable,.[1] with eigenbasis $\mathbf{x}_1, \ldots, \mathbf{x}_n$. Without loss of generality we can order the eigenvectors so that $P\mathbf{x}_i = 1\mathbf{x}_i$ for $1 \le i \le r$ and $P\mathbf{x}_i = 0\mathbf{x}_i$ for $r < i \le n$. If $X = \left(\mathbf{x}_1 \mid \cdots \mid \mathbf{x}_n \right)$ then we have

$$A = X \left(\begin{array}{c|c} I_r & O_{r,n-r} \\ \hline O_{n-r,r} & O_{n-r,n-r} \end{array} \right) X^{-1}.$$

Use this to prove that $P = AB^T$ for some $n \times r$ matrices A, B satisfying $B^T A = I_r$. [Hint: Let A be the first r columns of X and let B^T be the first r rows of X^{-1}.]

4. Normal Operators. Let V be a Hilbert space and let $A : V \to V$ be a continuous operator satisfying $A^*A = AA^*$. (If V is finite dimensional then we can view A^* as the conjugate transpose matrix.)

(a) Prove that $\langle A\mathbf{x}, A\mathbf{y}\rangle = \langle A^*\mathbf{x}, A^*\mathbf{y}\rangle$ for all $\mathbf{x}, \mathbf{y} \in V$.

(b) For all $\mathbf{x} \in V$ show that $A\mathbf{x} = \mathbf{0}$ if and only if $A^*\mathbf{x} = \mathbf{0}$. [Hint: Apply (a) with $\mathbf{x} = \mathbf{y}$.]

(c) Use (b) to show that $A\mathbf{x} = \lambda\mathbf{x}$ implies $A^*\mathbf{x} = \lambda^*\mathbf{x}$. [Hint: Consider the matrix $B = A - \lambda I$. Show that $B^*B = BB^*$ and then use part (b).]

(d) Suppose we have $A\mathbf{x} = \lambda\mathbf{x}$ and $A\mathbf{y} = \mu\mathbf{y}$ with $\lambda \ne \mu$. In this case, use part (c) to prove that $\langle \mathbf{x}, \mathbf{y}\rangle = 0$. [Hint: Show that $\lambda\langle \mathbf{x}, \mathbf{y}\rangle = \mu\langle \mathbf{x}, \mathbf{y}\rangle$.]

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[1] This follows from the fact that P satisfies the polynomial $f(x) = x(x-1)$, which has distinct roots.

5. Euler's Rotation Theorem. To be announced.

6. Gram-Schmidt Orthogonalization (Optional). Let V be an inner product space, possibly infinite dimensional. Given a basis $\mathbf{v}_1, \mathbf{v}_2, \ldots \in V$, we can create an orthonormal basis $\mathbf{u}_1, \mathbf{u}_2, \ldots \in V$ by the following recursive procedure:

- Let $\mathbf{u}_1 := \mathbf{v}_1$.
- For all $k \geq 1$, let $\mathbf{u}_{k+1} := \mathbf{v}_{k+1} - P_k(\mathbf{v}_{k+1})$, where $P_k : V \to V$ is the projection onto the subspace $U_k \subseteq V$ spanned by $\mathbf{u}_1, \ldots, \mathbf{u}_k$.

(a) The projection map $P_k : V \to V$ is defined by

$$P_k(\mathbf{v}) = \sum_{i=1}^{k} \frac{\langle \mathbf{u}_i, \mathbf{v} \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i.$$

Use this to show that $\mathbf{u}_{k+1}$ is orthogonal to each of the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$.

(b) Prove by induction that $\text{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_k\} = \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ for all $k \geq 1$.

(c) *Legendre Polynomials.* Consider the Hilbert space $L^2[-1, 1]$ with inner product

$$\langle f(x), g(x) \rangle = \int_{-1}^{1} f(x)g(x)\, dx.$$

Let $f_0(x), f_1(x), f_2(x), \ldots$ be the orthogonal basis obtained via the Gram-Schmidt process from the non-orthogonal basis $1, x, x^2, \ldots$.[2] Compute the first four polynomials:

$$f_0(x), f_1(x), f_2(x), f_3(x).$$

These polynomials arise in the study of the hydrogen atom.

[2] I started with index 0 instead of 1 so the polynomial $f_k(x)$ has degree k.