Math 510 **Fall 2022**

Homework 4 **Drew Armstrong**

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1. Alternating k-Forms. Let $\varphi : (\mathbb{R}^n)^k \to \mathbb{R}$ be any alternating k-form. We will write

$$\varphi(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k) = \varphi(A),$$

where A is the $n \times k$ matrix with columns $\mathbf{a}_1, \ldots, \mathbf{a}_k \in \mathbb{R}^n$.

(a) If A has a repeated column, prove that $\varphi(A) = 0$. [Hint: Without loss of generality, you can assume that $\mathbf{a}_1 = \mathbf{a}_2$. By assumption we have $\varphi(A') = -\varphi(A)$ where A' is obtained from A by swapping the first two columns.]

(b) If the columns of A are not independent, show that $\varphi(A) = 0$. [Hint: Without loss of generality, suppose that $\mathbf{a}_1 = b_2\mathbf{a}_2 + \cdots + b_n\mathbf{a}_n$ for some scalars $b_2, \ldots, b_n \in \mathbb{R}$. Now use part (a) and the fact that φ is linear in the first position.]

(c) If $k > n$, use part (b) to show that any alternating k-form on $\mathbb{R}^n$ must be the zero form, i.e., the form that sends every $n \times k$ matrix to zero.

Remark: It follows that $\dim \Lambda^k(\mathbb{R}^n) = 0$ for all $k > n$.

2. Volume of a k-Parallelogram in $\mathbb{R}^n$. For any k vectors $\mathbf{a}_1, \ldots, \mathbf{a}_k \in \mathbb{R}^n$ we define

$$\mathrm{Vol}_k(\mathbf{a}_1, \ldots, \mathbf{a}_k) = k\text{-volume of the } k\text{-parallelogram spanned by } \mathbf{a}_1, \ldots, \mathbf{a}_k \text{ in } \mathbb{R}^n.$$

If A is the $n \times k$ matrix with columns $\mathbf{a}_1, \ldots, \mathbf{a}_k \in \mathbb{R}^n$ we will also write

$$\mathrm{Vol}_k(A) = \mathrm{Vol}_k(\mathbf{a}_1, \ldots, \mathbf{a}_k).$$

When $k = n$, i.e., when A is square $n \times n$, we know from class that

$$\mathrm{Vol}_n(A) = |\det(A)|.$$

(a) If A is $n \times n$, use properties of determinants to show that $\mathrm{Vol}_n(A) = \sqrt{\det(A^TA)}$.

(b) Let A be 2×2 with columns $\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{R}^2$ and let θ_{12} be the angle between $\mathbf{a}_1$ and $\mathbf{a}_2$. Use part (a) to show that

$$\mathrm{Vol}_2(A) = \|\mathbf{a}_1\|\|\mathbf{a}_2\||\sin\theta_{12}|.$$

(c) Now let A be $n \times 2$ with columns $\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{R}^n$. For geometric reasons, we know that area of the 2-parallelogram spanned by $\mathbf{a}_1$ and $\mathbf{a}_2$ has the same formula as in part (b):

$$\mathrm{Vol}_2(A) = \|\mathbf{a}_1\|\|\mathbf{a}_2\||\sin\theta_{12}|.$$

Use this to prove that

$$\mathrm{Vol}_2(A) = \sqrt{\det(A^TA)},$$

even though the matrix A is not square, hence $\det(A)$ does not exist.

(d) Now let A be 3×3 with columns $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in \mathbb{R}^3$ and for all i, j let θ_{ij} be the angle between vectors $\mathbf{a}_i$ and $\mathbf{a}_i$. Use part (a) to show that $\mathrm{Vol}_3(A)$ equals

$$\|\mathbf{a}_1\|\|\mathbf{a}_2\|\|\mathbf{a}_3\|\sqrt{(1 + 2\cos\theta_{12}\cos\theta_{13}\cos\theta_{23} - (\cos^2\theta_{12} + \cos^2\theta_{13} + \cos^2\theta_{23}))}.$$

Since this formula can be expressed purely in terms of lengths and angles, it follows that $\mathrm{Vol}_3(A) = \sqrt{\det(A^TA)}$ for any $n \times 3$ matrix A, even though the determinant $\det(A)$ does not exist.

Remark: The same ideas show that $\mathrm{Vol}_k(A) = \sqrt{\det(A^TA)}$ for any $n \times k$ matrix A.