Math 510 Homework 1

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1. The Cauchy-Schwarz Inequality. Let V be an inner product space over $\mathbb{R}$. Prove that for all vectors $\mathbf{u}, \mathbf{v} \in V$ we have

$$|\langle \mathbf{u}, \mathbf{v} \rangle|^2 \leq \langle \mathbf{u}, \mathbf{u} \rangle \langle \mathbf{v}, \mathbf{v} \rangle.$$

[Hint: If $\mathbf{u} = \mathbf{0}$ and $\mathbf{v} = \mathbf{0}$ then it's easy, so let's assume that $\mathbf{v} \neq \mathbf{0}$. From Axiom (3d) we must have $\langle \mathbf{u} - t\mathbf{v}, \mathbf{u} - t\mathbf{v} \rangle \geq 0$ for any scalar $t \in \mathbb{R}$. Expand this expression using bilinearity and then substitute $t = \langle \mathbf{u}, \mathbf{v} \rangle / \langle \mathbf{v}, \mathbf{v} \rangle$.]

Proof. Consider vectors $\mathbf{u}, \mathbf{v} \in V$ with $\mathbf{v} \neq \mathbf{0}$, and a scalar $t \in \mathbb{R}$. Then we have

$$\begin{aligned} 0 &\leq \langle \mathbf{u} - t\mathbf{v}, \mathbf{u} - t\mathbf{v} \rangle \\ &= \langle \mathbf{u}, \mathbf{u} \rangle - 2t\langle \mathbf{u}, \mathbf{v} \rangle + t^2 \langle \mathbf{v}, \mathbf{v} \rangle. \end{aligned}$$

Since this holds for any $t \in \mathbb{R}$ we may substitute $t = \langle \mathbf{u}, \mathbf{v} \rangle / \langle \mathbf{v}, \mathbf{v} \rangle$ to get

$$\begin{aligned} 0 &\leq \langle \mathbf{u}, \mathbf{u} \rangle - 2 \cdot \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \cdot \langle \mathbf{u}, \mathbf{v} \rangle + \frac{\langle \mathbf{u}, \mathbf{v} \rangle^2}{\langle \mathbf{v}, \mathbf{v} \rangle^2} \cdot \langle \mathbf{v}, \mathbf{v} \rangle \\ 0 &\leq \langle \mathbf{u}, \mathbf{u} \rangle \langle \mathbf{v}, \mathbf{v} \rangle - 2\langle \mathbf{u}, \mathbf{v} \rangle^2 + \langle \mathbf{u}, \mathbf{v} \rangle^2 && \text{multiply by } \langle \mathbf{v}, \mathbf{v} \rangle \\ \langle \mathbf{u}, \mathbf{v} \rangle^2 &\leq \langle \mathbf{u}, \mathbf{u} \rangle \langle \mathbf{v}, \mathbf{v} \rangle. \end{aligned}$$

□

Remark: Here we have $|\langle \mathbf{u}, \mathbf{v} \rangle|^2 = \langle \mathbf{u}, \mathbf{v} \rangle^2$ because $\langle \mathbf{u}, \mathbf{v} \rangle$ is a real number. In the next proof we will allow $\langle \mathbf{u}, \mathbf{v} \rangle$ to be complex.

Proof for Hermitian Spaces. Let $\langle -, - \rangle$ be a Hermitian inner product on a complex vector space V. Then for any vectors $\mathbf{u}, \mathbf{v} \in V$ and for any complex scalar $t \in \mathbb{C}$ we have

$$\begin{aligned} 0 &\leq \langle \mathbf{u} - t\mathbf{v}, \mathbf{u} - t\mathbf{v} \rangle \\ &= \langle \mathbf{u}, \mathbf{u} \rangle - t\langle \mathbf{u}, \mathbf{v} \rangle - t^* \langle \mathbf{v}, \mathbf{u} \rangle + t^* t \langle \mathbf{v}, \mathbf{v} \rangle. \end{aligned}$$

Now suppose that $\mathbf{v} \neq \mathbf{0}$, so that $\langle \mathbf{v}, \mathbf{v} \rangle$ is a positive real number. Since the above inequality holds for any complex t we may substitute $t = \langle \mathbf{u}, \mathbf{v} \rangle^* / \langle \mathbf{v}, \mathbf{v} \rangle$ to get

$$\begin{aligned} 0 &\leq \langle \mathbf{u}, \mathbf{u} \rangle - t\langle \mathbf{u}, \mathbf{v} \rangle - t^* \langle \mathbf{v}, \mathbf{u} \rangle + t^* t \langle \mathbf{v}, \mathbf{v} \rangle \\ 0 &\leq \langle \mathbf{u}, \mathbf{u} \rangle - \frac{\langle \mathbf{u}, \mathbf{v} \rangle^*}{\langle \mathbf{v}, \mathbf{v} \rangle} \cdot \langle \mathbf{u}, \mathbf{v} \rangle - \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle^*} \cdot \langle \mathbf{v}, \mathbf{u} \rangle + \frac{\langle \mathbf{u}, \mathbf{v} \rangle \langle \mathbf{u}, \mathbf{v} \rangle^*}{\langle \mathbf{v}, \mathbf{v} \rangle^* \langle \mathbf{v}, \mathbf{v} \rangle} \cdot \langle \mathbf{v}, \mathbf{v} \rangle \\ 0 &\leq \langle \mathbf{u}, \mathbf{u} \rangle - \frac{\langle \mathbf{u}, \mathbf{v} \rangle^*}{\langle \mathbf{v}, \mathbf{v} \rangle} \cdot \langle \mathbf{u}, \mathbf{v} \rangle - \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \cdot \langle \mathbf{u}, \mathbf{v} \rangle^* + \frac{\langle \mathbf{u}, \mathbf{v} \rangle \langle \mathbf{u}, \mathbf{v} \rangle^*}{\langle \mathbf{v}, \mathbf{v} \rangle \langle \mathbf{v}, \mathbf{v} \rangle} \cdot \langle \mathbf{v}, \mathbf{v} \rangle \\ 0 &\leq \langle \mathbf{u}, \mathbf{u} \rangle - \frac{|\langle \mathbf{u}, \mathbf{v} \rangle|^2}{\langle \mathbf{v}, \mathbf{v} \rangle} - \frac{|\langle \mathbf{u}, \mathbf{v} \rangle|^2}{\langle \mathbf{v}, \mathbf{v} \rangle} + \frac{|\langle \mathbf{u}, \mathbf{v} \rangle|^2}{\langle \mathbf{v}, \mathbf{v} \rangle} \\ 0 &\leq \langle \mathbf{u}, \mathbf{u} \rangle \langle \mathbf{v}, \mathbf{v} \rangle - |\langle \mathbf{u}, \mathbf{v} \rangle|^2 \\ |\langle \mathbf{u}, \mathbf{v} \rangle|^2 &\leq \langle \mathbf{u}, \mathbf{u} \rangle \langle \mathbf{v}, \mathbf{v} \rangle. \end{aligned}$$

□

2. Normed Vector Spaces. Let V be an inner product space and consider the function

$$\|\mathbf{v}\| := \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}.$$

Use the axioms for inner products to prove the following properties.

(a) We have $\|\mathbf{v}\| \geq 0$ for all $\mathbf{v} \in V$, with $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = \mathbf{0}$.
(b) For all $a \in \mathbb{R}$ and $\mathbf{v} \in V$ we have $\|a\mathbf{v}\| = |a|\|\mathbf{v}\|$.
(c) For all $\mathbf{u}, \mathbf{v} \in V$ we have $\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\|+\|\mathbf{v}\|$. [Hint: Expand $\|\mathbf{u}+\mathbf{v}\|^2 = \langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}\rangle$ and use the Cauchy-Schwarz inequality to show that $\|\mathbf{u}+\mathbf{v}\|^2 \leq (\|\mathbf{u}\| + \|\mathbf{v}\|)^2$.]

I will prove these for the slightly more general case of Hermitian inner products.

(a): Since $\langle \mathbf{v}, \mathbf{v}\rangle \geq 0$ we have $\sqrt{\langle \mathbf{v}, \mathbf{v}\rangle} \geq 0$. Furthermore, we have

$$\sqrt{\langle \mathbf{v}, \mathbf{v}\rangle} = 0 \quad \Longleftrightarrow \quad \langle \mathbf{v}, \mathbf{v}\rangle = 0 \quad \Longleftrightarrow \quad \mathbf{v} = \mathbf{0}.$$

(b): For all $\alpha \in \mathbb{C}$ and $\mathbf{v} \in V$ we have

$$\|\alpha\mathbf{v}\|^2 = \langle \alpha\mathbf{v}, \alpha\mathbf{v}\rangle = \alpha^*\alpha\langle \mathbf{v}, \mathbf{v}\rangle = |\alpha|^2\|\mathbf{v}\|^2.$$

Then taking the square root of each side gives $\|\alpha\mathbf{v}\| = |\alpha|\|\mathbf{v}\|$.

(c): From the Cauchy-Schwarz inequality we have $|\langle \mathbf{u}, \mathbf{v}\rangle|^2 \leq \|\mathbf{u}\|^2\|\mathbf{v}\|^2$, and hence $|\langle \mathbf{u}, \mathbf{v}\rangle| \leq \|\mathbf{u}\|\|\mathbf{v}\|$. Now consider any complex number $\alpha = a + ib \in \mathbb{C}$. By thinking of the right triangle in $\mathbb{R}^2$ with side lengths $|a|, |b|, \sqrt{a^2+b^2}$ we see that

$$\mathrm{Re}(\alpha) \leq |\mathrm{Re}(\alpha)| = |a| \leq \sqrt{a^2+b^2} = |\alpha|.$$

In particular, for any vectors $\mathbf{u}, \mathbf{v} \in V$ we have $\mathrm{Re}(\langle \mathbf{u}, \mathbf{v}\rangle) \leq |\langle \mathbf{u}, \mathbf{v}\rangle|$. Combining this observation with Cauchy Schwarz gives

$$\begin{aligned}
\|\mathbf{u}+\mathbf{v}\|^2 &= \langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}\rangle \\
&= \langle \mathbf{u}, \mathbf{u}\rangle + \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{v}, \mathbf{u}\rangle + \langle \mathbf{v}, \mathbf{v}\rangle \\
&= \langle \mathbf{u}, \mathbf{u}\rangle + \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{v}\rangle^* + \langle \mathbf{v}, \mathbf{v}\rangle \\
&= \langle \mathbf{u}, \mathbf{u}\rangle + 2\cdot\mathrm{Re}(\langle \mathbf{u}, \mathbf{v}\rangle) + \langle \mathbf{v}, \mathbf{v}\rangle \\
&\leq \langle \mathbf{u}, \mathbf{u}\rangle + 2\cdot|\langle \mathbf{u}, \mathbf{v}\rangle| + \langle \mathbf{v}, \mathbf{v}\rangle \\
&\leq \langle \mathbf{u}, \mathbf{u}\rangle + 2\|\mathbf{u}\|\|\mathbf{v}\| + \langle \mathbf{v}, \mathbf{v}\rangle \\
&= \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\| + \|\mathbf{v}\|^2 \\
&= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2.
\end{aligned}$$

Finally, we take square roots to obtain $\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$. □

3. Orthonormal Sets of Vectors. Let V be an inner product space. Suppose that a set of vectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n \in V$ satisfies

$$\langle \mathbf{b}_i, \mathbf{b}_j\rangle = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

In this case we say that the vectors are *orthonormal*.

(a) If $\mathbf{v} = a_1\mathbf{b}_1 + \cdots + a_n\mathbf{b}_n$, show that $a_i = \langle \mathbf{v}, \mathbf{b}_i\rangle$ for all i.
(b) Use part (a) to show that the set $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n$ is linearly independent.
(c) If $\mathbf{v} = a_1\mathbf{b}_1 + \cdots + a_n\mathbf{b}_n$, show that $\|\mathbf{v}\|^2 = a_1^2 + \cdots + a_n^2$.[1]

[1] Define $\|\mathbf{v}\|^2 = \langle \mathbf{v}, \mathbf{v}\rangle$ as in Problem 2.

I will prove these in the slightly more general case of Hermitian inner products.

(a): Suppose that $\langle \mathbf{b}_i, \mathbf{b}_j \rangle = \delta_{ij}$ and $\mathbf{v} = a_1\mathbf{b}_1 + \cdots + a_n\mathbf{b}_n$. Then we have

$$\begin{aligned}
\langle \mathbf{b}_i, \mathbf{v} \rangle &= \langle \mathbf{b}_i, a_1\mathbf{b}_1 + \cdots + a_n, \mathbf{b}_j \rangle \\
&= a_1\langle \mathbf{b}_i, \mathbf{b}_1 \rangle + \cdots + a_n\langle \mathbf{b}_i, \mathbf{b}_n \rangle \\
&= a_1\langle \mathbf{b}_i, \mathbf{b}_1 \rangle + \cdots + a_i\langle \mathbf{b}_i, \mathbf{b}_i \rangle + \cdots + a_n\langle \mathbf{b}_i, \mathbf{b}_n \rangle \\
&= 0a_1 + \cdots + 0a_{i-1} + 1a_i + 0a_{i+1} + \cdots + 0a_n \\
&= a_i.
\end{aligned}$$

Alternatively, we can use symbolic notation:

$$\langle \mathbf{b}_i, \mathbf{v} \rangle = \langle \mathbf{b}_i, \sum_j a_j\mathbf{b}_j \rangle = \sum_j a_i\langle \mathbf{b}_i, \mathbf{b}_j \rangle = \sum_j a_j\delta_{ij} = a_i.$$

Remark: If the coefficients are complex then we also have $\langle \mathbf{v}, \mathbf{b}_i \rangle = a_i^*$.

(b): Suppose that we have $a_1\mathbf{b}_1 + \cdots + a_n\mathbf{b}_n = \mathbf{0}$ for some constants $a_1, \ldots, a_n \in \mathbb{C}$. Then for all i, we have from part (a) that

$$a_i = \langle \mathbf{b}_i, \mathbf{0} \rangle = 0.$$

Hence the set $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is linearly independent. Remark: In fact, we only needed to assume that $\langle \mathbf{b}_i, \mathbf{b}_j \rangle = 0$ for all $i \neq j$. The specific values of $\langle \mathbf{b}_i, \mathbf{b}_i \rangle$ are not important. In summary:

Every orthogonal set of vectors is independent.

(c): This time I will just use the symbolic notation. If you don't follow the steps, try writing out a long form proof as in part (a). Let $\mathbf{v} = \sum_i a_i\mathbf{b}_i$ where $\langle \mathbf{b}_i, \mathbf{b}_j \rangle = \delta_{ij}$. Then we have

$$\begin{aligned}
\|\mathbf{v}\|^2 &= \langle \mathbf{v}, \mathbf{v} \rangle \\
&= \langle \sum_i a_i\mathbf{b}_i, \sum_j a_j\mathbf{b}_j \rangle \\
&= \sum_{i,j} a_i^* a_j \langle \mathbf{b}_i, \mathbf{b}_j \rangle \\
&= \sum_{i,j} a_i^* a_j \delta_{i,j} \\
&= \sum_i a_i^* a_i \\
&= \sum_i |a_i|^2.
\end{aligned}$$

Remark: If the coefficients a_i are real then we have $|a_i|^2 = a_i^2$ and hence $\|\mathbf{v}\|^2 = \sum_i a_i^2$.

4. Fourier Series. Consider the space $L^2[0,1]$ of functions[2] $[0,1] \to \mathbb{R}$ with inner product

$$\langle f(x), g(x) \rangle = \int_0^1 f(x)g(x)\,dx.$$

[2] We require that $\int_0^1 f(x)^2\,dx$ exists and is finite.

For any integer $n \geq 1$ we define the functions $s_n(x) := \sqrt{2}\sin(2\pi nx)$ and $c_n(x) := \sqrt{2}\cos(2\pi nx)$. Recall the trigonometric angle sum identities:

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta,$$
$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta.$$

(a) Prove that $\langle 1, s_n(x)\rangle = \langle 1, c_n(x)\rangle = 0$ for all n.
(b) Use the angle sum identities to prove that

$$2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta),$$
$$2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta),$$
$$2\cos\alpha\cos\beta = \cos(\alpha-\beta) + \cos(\alpha+\beta).$$

(c) Use (b) to prove that $\langle s_m(x), c_n(x)\rangle = 0$ for all $m, n \geq 1$.
(d) Use (b) to prove that $\langle s_m(x), s_n(x)\rangle = \delta_{mn}$.
(e) Use (b) to prove that $\langle c_m(x), c_n(x)\rangle = \delta_{mn}$.

(a): For any $n \in \mathbb{Z}$ we have $\cos(2\pi n) = 1$ and hence

$$\begin{aligned}\langle 1, s_n(x)\rangle &= \sqrt{2}\int_0^1 1\sin(2\pi nx)\,dx \\ &= -\frac{\sqrt{2}}{2\pi n}\left[-\cos(2\pi nx)\right]_0^1 \\ &= -\frac{\sqrt{2}}{2\pi}\left[-\cos(2\pi n) + \cos(0)\right] \\ &= -\frac{\sqrt{2}}{2\pi n}\left[-1+1\right] \\ &= 0.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\langle 1, c_n(x)\rangle &= \sqrt{2}\int_0^1 1\cos(2\pi nx)\,dx \\ &= \frac{\sqrt{2}}{2\pi n}\left[\sin(2\pi nx)\right]_0^1 \\ &= \frac{\sqrt{2}}{2\pi}\left[\sin(2\pi n) - \sin(0)\right] \\ &= \frac{\sqrt{2}}{2\pi n}\left[0-0\right] \\ &= 0.\end{aligned}$$

(b): Add the relevant angle sum/difference formulas.

(c): For all $m, n \in \mathbb{Z}$, part (b) tells us that

$$2\sin(2\pi mx)\cos(2\pi nx) = \sin(2\pi(m+n)x) + \sin(2\pi(m-n)x).$$

If $m = n$ then this gives

$$\begin{aligned}
\langle s_m(x), c_n(x) \rangle &= \langle s_n(x), c_n(x) \rangle \\
&= \int_0^1 2\sin(2\pi nx)\cos(2\pi nx)\,dx \\
&= \int_0^1 [\sin(2\pi(n+n)x) + \sin(2\pi(n-n)x)]\,dx \\
&= \int_0^1 [\sin(2\pi(n+n)x) + 0]\,dx \\
&= \left[-\frac{1}{2\pi(n+n)x}\cos(2\pi(n+n)x)\right]_0^1 \\
&= \left[-\frac{1}{2\pi(n+n)x}(1-1)\right] \\
&= 0.
\end{aligned}$$

And if $m \neq n$ then we have

$$\begin{aligned}
\langle s_m(x), c_n(x) \rangle &= \int_0^1 2\sin(2\pi mx)\cos(2\pi nx)\,dx \\
&= \int_0^1 [\sin(2\pi(m+n)x) + \sin(2\pi(m-n)x)]\,dx \\
&= \left[-\frac{1}{2\pi(m+n)x}\cos(2\pi(m+n)x) - \frac{1}{2\pi(m-n)}\cos(2\pi(m-n)x)\right]_0^1 \\
&= \left[-\frac{1}{2\pi(m+n)x}(1-1) - \frac{1}{2\pi(m-n)}(1-1)\right] \\
&= 0.
\end{aligned}$$

(d): If $m = n$ then we have

$$\begin{aligned}
\langle s_m(x), s_n(x) \rangle &= \langle s_n(x), s_n(x) \rangle \\
&= \int_0^1 2\sin(2\pi nx)\sin(2\pi nx)\,dx \\
&= \int_0^1 [\cos(2\pi(n-n)x) - \cos(2\pi(n+n)x)]\,dx \\
&= \int_0^1 [1 - \cos(2\pi(n+n)x)]\,dx \\
&= \left[x - \frac{1}{2\pi(n+n)}\sin(2\pi(n+n)x)\right]_0^1 \\
&= [(1-0) - (0-0)] \\
&= 1.
\end{aligned}$$

And if $m \neq n$ then we have

$$\begin{aligned}
\langle s_m(x), s_n(x)\rangle &= \int_0^1 2\sin(2\pi m x)\sin(2\pi n x)\,dx \\
&= \int_0^1 \left[\cos(2\pi(m-n)x) - \cos(2\pi(m+n)x)\right]\,dx \\
&= \left[\frac{1}{2\pi(m-n)}\sin(2\pi(m-n)x) - \frac{1}{2\pi(m+n)}\sin(2\pi(m+n)x)\right]_0^1 \\
&= \left[\frac{1}{2\pi(m-n)}(0-0) - \frac{1}{2\pi(m+n)}(0-0)\right] \\
&= 0.
\end{aligned}$$

(e): If $m = n$ then we have

$$\begin{aligned}
\langle c_m(x), c_n(x)\rangle &= \langle c_n(x), c_n(x)\rangle \\
&= \int_0^1 2\cos(2\pi n x)\cos(2\pi n x)\,dx \\
&= \int_0^1 \left[\cos(2\pi(n-n)x) + \cos(2\pi(n+n)x)\right]\,dx \\
&= \int_0^1 \left[1 + \cos(2\pi(n+n)x)\right]\,dx \\
&= \left[x + \frac{1}{2\pi(n+n)}\sin(2\pi(n+n)x)\right]_0^1 \\
&= \left[(1-0) + (0-0)\right] \\
&= 1.
\end{aligned}$$

And if $m \neq n$ then we have

$$\begin{aligned}
\langle c_m(x), c_n(x)\rangle &= \int_0^1 2\cos(2\pi m x)\cos(2\pi n x)\,dx \\
&= \int_0^1 \left[\cos(2\pi(m-n)x) + \cos(2\pi(m+n)x)\right]\,dx \\
&= \left[\frac{1}{2\pi(m-n)}\sin(2\pi(m-n)x) + \frac{1}{2\pi(m+n)}\sin(2\pi(m+n)x)\right]_0^1 \\
&= \left[\frac{1}{2\pi(m-n)}(0-0) + \frac{1}{2\pi(m+n)}(0-0)\right] \\
&= 0.
\end{aligned}$$

Remark: Those calculations are certainly annoying, and I'm certain I made several typos. Complex Fourier series make the calculations much easier. In that case, for each integer $n \in \mathbb{Z}$ we define the function $\chi_n(x) = e^{i2\pi n x}$ from the real interval $[0,1]$ to the complex numbers $\mathbb{C}$. We view these as elements of the space $L^2[0,1]$ with Hermitian inner product

$$\langle f(x), g(x)\rangle = \int_0^1 f(x)^* g(x)\,dx.$$

I claim that $\langle \chi_m(x), \chi_n(x) \rangle = \delta_{mn}$. Indeed, if $m = n$ then we have

$$\langle \chi_m(x), \chi_n(x) \rangle = \langle \chi_n(x), \chi_n(x) \rangle = \int_0^1 e^{-2\pi nx} e^{i2\pi nx}\, dx = \int_0^1 1\, dx = 1.$$

And if $m \neq n$ then we have

$$\begin{aligned}
\langle \chi_m(x), \chi_n(x) \rangle &= \int_0^1 (e^{i2\pi mx})^* e^{i2\pi nx}\, dx \\
&= \int_0^1 e^{-i2\pi mx} e^{i2\pi nx}\, dx \\
&= \int_0^1 e^{i2\pi(n-m)x}\, dx \\
&= \frac{1}{i2\pi(n-m)} \left[e^{i2\pi(n-m)x} \right]_0^1 \\
&= \frac{1}{i2\pi(n-m)} [1 - 1] \\
&= 0.
\end{aligned}$$

Wasn't that easier?

5. Steinitz Exchange (Optional). Let I and S be finite subsets of a vector space V, where I is an independent set and S is a spanning set. Let's say $\#I = m$ and $\#S = n$. Our goal is to show that $m \leq n$. To prove this, we will use the method of Steinitz (1913). For any $1 \leq k \leq \min\{m, n\}$ consider the following statement:

$P(k)$: *For any k elements $\mathbf{u}_1, \ldots, \mathbf{u}_k \in I$, there exist some $n-k$ elements $\mathbf{v}_1, \ldots, \mathbf{v}_{n-k} \in S$ such that the set $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{v}_1, \ldots, \mathbf{v}_{n-k}\}$ spans all of V.*

(a) Prove that $P(1)$ is a true statement. [Hint: Write $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and choose any vector $\mathbf{u} \in I$. Since S spans V we can write $\mathbf{u} = \sum a_i \mathbf{v}_i$, and since $\mathbf{u} \neq \mathbf{0}$ we must have $a_p \neq 0$ for some p. Show that $\{\mathbf{u}, \mathbf{v}_1, \ldots, \mathbf{v}_{p-1}, \mathbf{v}_p, \ldots, \mathbf{v}_n\}$ is a spanning set.]

(b) Assume that $P(k)$ is true for some $1 \leq k < \min\{m, n\}$. In this case prove that $P(k+1)$ is also true. [Hint: Choose any $\mathbf{u}_1, \ldots, \mathbf{u}_{k+1} \in I$. Since $P(k)$ is true we can find $\mathbf{v}_1, \ldots, \mathbf{v}_{n-k} \in S$ such that $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{v}_1, \ldots, \mathbf{v}_{n-k}\}$ spans V. In particular we can write $\mathbf{u}_{k+1} = \sum b_i \mathbf{u}_i + \sum a_i \mathbf{v}_i$. By the independence of I we must have $a_p \neq 0$ for some p. Show that $\{\mathbf{u}_1, \ldots, \mathbf{u}_{k+1}, \mathbf{v}_1, \ldots, \mathbf{v}_{p-1}, \mathbf{v}_{p+1}, \ldots, \mathbf{v}_{n-k}\}$ spans V.]

(c) It follows from (a) and (b) that $P(k)$ is true for all $1 \leq k \leq \min\{m, n\}$. Use this fact to prove that $m \leq n$. [Hint: Write $I = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$. If $m > n$ then taking $k = n$ shows that $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is a spanning set. But then we can write $\mathbf{u}_{n+1} = \sum a_i \mathbf{u}_i$, which contradicts the fact that I is independent.]

(d) Use part (c) to prove that any two bases for V have the same number of elements.

Remark: There is another way to prove this using matrix arithmetic, which will seem easier when we get there, but which is ultimately a much longer proof.

(a): Consider any vector $\mathbf{u} \in I$. Since the set S spans V there exist scalars a_i such that $\mathbf{u} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$, and since $\mathbf{u} \neq \mathbf{0}$ at least one of these scalars is nonzero. Let's say $a_p \neq 0$. Then we can solve for $\mathbf{v}_p$ as follows:

$$\mathbf{v}_p = \frac{1}{a_p}\mathbf{u} + \frac{-a_1}{a_p}\mathbf{v}_1 + \cdots + \frac{-a_{p-1}}{a_p}\mathbf{v}_{p-1} + \frac{-a_{p+1}}{a_p}\mathbf{v}_{p+1} + \cdots + \frac{-a_n}{a_p}\mathbf{v}_n.$$

To show that $\{\mathbf{u}, \mathbf{v}_1, \ldots, \mathbf{v}_{p-1}, \mathbf{v}_{p+1}, \ldots, \mathbf{v}_n\}$ is a spanning set, consider any $\mathbf{x} \in V$. Since S is spanning there exist some coefficients $d_1, \ldots, d_n$ such that

$$\mathbf{x} = d_1\mathbf{v}_1 + \cdots + d_p\mathbf{v}_p + \cdots + d_n\mathbf{v}_n.$$

Then substituting our previous expression for $\mathbf{v}_p$ gives

$$\mathbf{x} = \frac{d_p}{a_p}\mathbf{u} + \left(d_1 - \frac{d_p a_1}{a_p}\right)\mathbf{v}_1 + \cdots + \left(d_{p-1} - \frac{d_p a_{p-1}}{a_p}\right)\mathbf{v}_{p-1} + \left(d_{p+1} - \frac{d_p a_{p+1}}{a_p}\right)\mathbf{v}_{p+1} + \cdots + \left(d_n - \frac{d_p a_n}{a_p}\right)\mathbf{v}_n,$$

which shows that $\mathbf{x}$ is in the span of $\{\mathbf{u}, \mathbf{v}_1, \ldots, \mathbf{v}_{p-1}, \mathbf{v}_{p+1}, \ldots, \mathbf{v}_n\}$. □

(b): Now fix some $1 \leq k < \min\{m, n\}$ and assume for induction that $P(k)$ is true. In order to prove that $P(k+1)$ is also true, we consider any set of $k+1$ elements: $\mathbf{u}_1, \ldots, \mathbf{u}_{k+1} \in I$. Applying the statement $P(k)$ to the subset $\mathbf{u}_1, \ldots, \mathbf{u}_k$ tells us that there exist some $\mathbf{v}_1, \ldots, \mathbf{v}_{n-k} \in S$ such that the set $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{v}_1, \ldots, \mathbf{v}_{n-k}\}$ spans all of V. In particular, we can express $\mathbf{u}_{k+1}$ as a linear combination

$$\mathbf{u}_{k+1} = b_1\mathbf{u}_1 + \cdots + \mathbf{b}_k\mathbf{u}_k + a_1\mathbf{v}_1 + \cdots + a_{n-k}\mathbf{v}_{n-k}$$

for some scalars $b_1, \ldots, b_k$ and $a_1, \ldots, a_{n-k}$. Observe that not all of the coefficients a_i are zero, since otherwise this would give a nontrivial linear relation

$$\mathbf{u}_{k+1} = b_1\mathbf{u}_1 + \cdots + \mathbf{b}_k\mathbf{u}_k,$$

contradicting the fact that the $\mathbf{u}_i$ are independent. To be specific, let's say that $a_p \neq 0$. Then we can solve for $\mathbf{v}_p$ in terms of $\mathbf{u}_1, \ldots, \mathbf{u}_{k+1}, \mathbf{v}_1, \ldots, \mathbf{v}_{p-1}, \mathbf{v}_{p+1}, \ldots, \mathbf{v}_{n-k}$:

$$(*) \quad \mathbf{v}_p = \frac{-b_1}{a_p}\mathbf{u}_1 + \cdots + \frac{-b_k}{a_p}\mathbf{u}_k + \frac{1}{a_p}\mathbf{u}_{k+1} + \left(d_1 - \frac{d_p a_1}{a_p}\right)\mathbf{v}_1 + \cdots + \frac{-a_{p-1}}{a_p}\mathbf{v}_{p-1} + \frac{-a_{p+1}}{a_p}\mathbf{v}_{p+1} + \cdots + \frac{-a_{n-k}}{a_p}\mathbf{v}_{n-k}.$$

Finally, I claim that the set $\{\mathbf{u}_1, \ldots, \mathbf{u}_{k+1}, \mathbf{v}_1, \ldots, \mathbf{v}_{p-1}, \mathbf{v}_{p+1}, \ldots, \mathbf{v}_{n-k}\}$ spans all of V. Indeed, for any $\mathbf{x} \in V$ we have assumed the existence of coefficients $c_1, \ldots, c_k, d_1, \ldots, d_{n-k}$ such that

$$\mathbf{x} = c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k + d_1\mathbf{v}_1 + \cdots + d_p\mathbf{v}_p + \cdots + d_{n-k}\mathbf{v}_{n-k}.$$

Substituting the expression $(*)$ gives

$$\mathbf{x} = \left(c_1 - \frac{d_p b_1}{a_p}\right)\mathbf{u}_1 + \cdots + \left(c_k - \frac{d_p b_k}{a_p}\right)\mathbf{u}_k + \frac{d_p}{a_p}\mathbf{u}_{k+1} + \left(d_1 - \frac{d_p a_1}{a_p}\right)\mathbf{v}_1 + \cdots + \left(d_{p-1} - \frac{d_p a_{p-1}}{a_p}\right)\mathbf{v}_{p-1} + \left(d_{p+1} - \frac{d_p a_{p+1}}{a_p}\right)\mathbf{v}_{p+1} + \cdots + \left(d_{n-k} - \frac{d_p a_{n-k}}{a_p}\right)\mathbf{v}_{n-k},$$

which shows that $\mathbf{x}$ is in the span of $\{\mathbf{u}_1, \ldots, \mathbf{u}_{k+1}, \mathbf{v}_1, \ldots, \mathbf{v}_{p-1}, \mathbf{v}_{p+1}, \ldots, \mathbf{v}_{n-k}\}$. □

(c): It follows from (a) and (b) that $P(k)$ is true for all $1 \leq k \leq \min\{m, n\}$. We will use this to prove that $m \leq n$. Indeed, suppose for contradiction that $m \geq n+1$, so that $\min\{m, n\} = n$. Taking $k = n$ tells us that $P(n)$ is a true statement, which means that $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is a spanning set. But since $m \geq n+1$ there exists another vector $\mathbf{u}_{n+1}$, and since $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$

is a spanning set we can write $\mathbf{u}_{n+1} = a_1\mathbf{u}_1 + \cdots + a_n\mathbf{u}_n$. This nontrivial relation contradicts the fact that I is independent.

(d): Combining (a), (b) and (c) we have shown that for any independent set I and spanning set S we must have $\#I \leq \#S$. Now suppose that B_1, B_2 are two bases for V. Since B_1 is independent and B_2 is spanning we have $\#B_1 \leq \#B_2$. On the other hand, since B_2 is independent and B_1 is spanning we have $\#B_2 \leq \#B_1$. It follows that $\#B_1 = \#B_2$ as desired.