Math 510 **Fall 2022**
Homework 1 **Drew Armstrong**

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1. The Cauchy-Schwarz Inequality. Let V be an inner product space over $\mathbb{R}$. Prove that for all vectors $\mathbf{u},\mathbf{v}\in V$ we have

$$|\langle \mathbf{u},\mathbf{v}\rangle|^2 \leq \langle \mathbf{u},\mathbf{u}\rangle\langle \mathbf{v},\mathbf{v}\rangle.$$

[Hint: If $\mathbf{u}=\mathbf{0}$ and $\mathbf{v}=\mathbf{0}$ then it's easy, so let's assume that $\mathbf{v}\neq\mathbf{0}$. From Axiom (3d) we must have $\langle \mathbf{u}+t\mathbf{v},\mathbf{u}+t\mathbf{v}\rangle\geq 0$ for any scalar $t\in\mathbb{R}$. Expand this expression using bilinearity and then substitute $t=\langle \mathbf{u},\mathbf{v}\rangle/\langle \mathbf{v},\mathbf{v}\rangle$.]

2. Normed Vector Spaces. Let V be an inner product space and consider the function

$$\|\mathbf{v}\| := \sqrt{\langle \mathbf{v},\mathbf{v}\rangle}.$$

Use the axioms for inner products to prove the following properties.

(a) We have $\|\mathbf{v}\|\geq 0$ for all $\mathbf{v}\in V$, with $\|\mathbf{v}\|=0$ if and only if $\mathbf{v}=\mathbf{0}$.
(b) For all $a\in\mathbb{R}$ and $\mathbf{v}\in V$ we have $\|a\mathbf{v}\|=|a|\|\mathbf{v}\|$.
(c) For all $\mathbf{u},\mathbf{v}\in V$ we have $\|\mathbf{u}+\mathbf{v}\|\leq\|\mathbf{u}\|+\|\mathbf{v}\|$. [Hint: Expand $\|\mathbf{u}+\mathbf{v}\|^2=\langle \mathbf{u}+\mathbf{v},\mathbf{u}+\mathbf{v}\rangle$ and use the Cauchy-Schwarz inequality to show that $\|\mathbf{u}+\mathbf{v}\|^2\leq(\|\mathbf{u}\|+\|\mathbf{v}\|)^2$.]

3. Orthonormal Sets of Vectors. Let V be an inner product space. Suppose that a set of vectors $\mathbf{b}_1,\mathbf{b}_2,\ldots,\mathbf{b}_n\in V$ satisfies

$$\langle \mathbf{b}_i,\mathbf{b}_j\rangle = \begin{cases} 1 & i=j, \\ 0 & i\neq j. \end{cases}$$

In this case we say that the vectors are *orthonormal*.

(a) If $\mathbf{v}=a_1\mathbf{b}_1+\cdots+a_n\mathbf{b}_n$, show that $a_i=\langle \mathbf{v},\mathbf{b}_i\rangle$ for all i.
(b) Use part (a) to show that the set $\mathbf{b}_1,\mathbf{b}_2,\ldots,\mathbf{b}_n$ is linearly independent.
(c) If $\mathbf{v}=a_1\mathbf{b}_1+\cdots+a_n\mathbf{b}_n$, show that $\|\mathbf{v}\|^2=a_1^2+\cdots+a_n^2$.[1]

4. Fourier Series. Consider the space $L^2[0,1]$ of functions[2] $[0,1]\to\mathbb{R}$ with inner product

$$\langle f(x),g(x)\rangle = \int_0^1 f(x)g(x)\,dx.$$

For any integer $n\geq 1$ we define the functions $s_n(x):=\sqrt{2}\sin(2\pi nx)$ and $c_n(x):=\sqrt{2}\cos(2\pi nx)$. Recall the trigonometric angle sum identities:

$$\cos(\alpha\pm\beta)=\cos\alpha\cos\beta\mp\sin\alpha\sin\beta,$$
$$\sin(\alpha\pm\beta)=\sin\alpha\cos\beta\pm\cos\alpha\sin\beta.$$

(a) Prove that $\langle 1,s_n(x)\rangle=\langle 1,c_n(x)\rangle=0$ for all n.
(b) Use the angle sum identities to prove that

$$2\sin\alpha\cos\beta=\sin(\alpha+\beta)+\sin(\alpha-\beta),$$
$$2\sin\alpha\sin\beta=\cos(\alpha-\beta)-\cos(\alpha+\beta),$$
$$2\cos\alpha\cos\beta=\cos(\alpha-\beta)+\cos(\alpha+\beta).$$

(c) Use (b) to prove that $\langle s_m(x),c_n(x)\rangle=0$ for all $m,n\geq 1$.

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[1] Define $\|\mathbf{v}\|^2=\langle \mathbf{v},\mathbf{v}\rangle$ as in Problem 2.
[2] We require that $\int_0^1 f(x)^2\,dx$ exists and is finite.

(d) Use (b) to prove that $\langle s_m(x), s_n(x) \rangle = \delta_{mn}$.
(e) Use (b) to prove that $\langle c_m(x), c_n(x) \rangle = \delta_{mn}$.

5. Steinitz Exchange (Optional). Let I and S be finite subsets of a vector space V, where I is an independent set and S is a spanning set. Let's say $\#I = m$ and $\#S = n$. Our goal is to show that $m \leq n$. To prove this, we will use the method of Steinitz (1913). For any $1 \leq k \leq \min\{m, n\}$ consider the following statement:

$P(k)$: *For any k elements $\mathbf{u}_1, \ldots, \mathbf{u}_k \in I$, there exist some $n-k$ elements $\mathbf{v}_1, \ldots, \mathbf{v}_{n-k} \in S$ such that the set $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{v}_1, \ldots, \mathbf{v}_{n-k}\}$ spans all of V.*

(a) Prove that $P(1)$ is a true statement. [Hint: Write $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and choose any vector $\mathbf{u} \in I$. Since S spans V we can write $\mathbf{u} = \sum a_i \mathbf{s}_i$, and since $\mathbf{u} \neq \mathbf{0}$ we must have $a_p \neq 0$ for some p. Show that $\{\mathbf{u}, \mathbf{v}_1, \ldots, \mathbf{v}_{p-1}, \mathbf{v}_p, \ldots, \mathbf{v}_n\}$ is a spanning set.]
(b) Assume that $P(k)$ is true for some $1 \leq k < \min\{m, n\}$. In this case prove that $P(k+1)$ is also true. [Hint: Choose any $\mathbf{u}_1, \ldots, \mathbf{u}_{k+1} \in I$. Since $P(k)$ is true we can find $\mathbf{v}_1, \ldots, \mathbf{v}_{n-k} \in S$ such that $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{v}_1, \ldots, \mathbf{v}_{n-k}\}$ spans V. In particular we can write $\mathbf{u}_{k+1} = \sum b_i \mathbf{u}_i + \sum a_i \mathbf{v}_i$. By the independence of I we must have $a_p \neq 0$ for some p. Show that $\{\mathbf{u}_1, \ldots, \mathbf{u}_{k+1}, \mathbf{v}_1, \ldots, \mathbf{v}_{p-1}, \mathbf{v}_{p+1}, \ldots, \mathbf{v}_{n-k}\}$ spans V.]
(c) It follows from (a) and (b) that $P(k)$ is true for all $1 \leq k \leq \min\{m, n\}$. Use this fact to prove that $m \leq n$. [Hint: Write $I = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$. If $m > n$ then taking $k = n$ shows that $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is a spanning set. But then we can write $\mathbf{u}_{n+1} = \sum a_i \mathbf{u}_i$, which contradicts the fact that I is independent.]
(d) Use part (c) to prove that any two bases for V have the same number of elements.

Remark: There is another way to prove this using matrix arithmetic, which will seem easier when we get there, but which is ultimately a much longer proof.