Math 510 **Fall 2022**
Review for Exam 2 **Drew Armstrong**

- State the definitions of row space, column space, nullspace. It is important to know that

$$\mathcal{C}(A) = \text{all vectors of the form } A\mathbf{x},$$

and

$$A\mathbf{x} = \mathbf{0} \iff \mathbf{x} \text{ is orthogonal to every row of } A.$$

- The previous observation says that $\mathcal{N}(A) = \mathcal{R}(A)^\perp$. In general, if $U \subseteq V$ is a subspace of an inner product space V then we define

$$U^\perp = \{\mathbf{v} \in V : \langle \mathbf{u}, \mathbf{v} \rangle = 0 \text{ for all } \mathbf{u} \in U\}.$$

If V is finite dimensional, then you should know (but you don't need to prove) that

$$\dim U + \dim U^\perp = \dim V.$$

- If A is $m \times n$ then $\mathcal{R}(A)$ and $\mathcal{N}(A)$ are subspaces of $\mathbb{R}^n$, while $\mathcal{C}(A)$ and $\mathcal{N}(A^T)$ are subspaces of $\mathbb{R}^m$. It follows from the previous fact that

$$\dim \mathcal{R}(A) + \dim \mathcal{N}(A) = n \quad \text{and} \quad \dim \mathcal{C}(A) + \dim \mathcal{N}(A^T) = m.$$

- The Fundamental Theorem says that $\dim \mathcal{R}(A) = \dim \mathcal{C}(A)$. You don't need to prove this. We define

$$\text{rank}(A) := \dim \mathcal{R}(A) = \dim \mathcal{C}(A).$$

- Know that row operations correspond to multiplying EA with E elementary, and column operations correspond to multiplying AF with F elementary. Given a small invertible matrix A, be able to express A as a product of elementary matrices.

- Given specific A, find bases for the four fundamental subspaces. Method: Compute RREF of A. The nonzero rows of the RREF are a basis for $\mathcal{R}(A)$. The pivot columns of the RREF are not a basis for $\mathcal{C}(A)$, but the corresponding columns in A are a basis for $\mathcal{C}(A)$. Alternatively, the nonzero rows of the RREF of A^T are a basis for $\mathcal{C}(A)$. To compute $\mathcal{N}(A)$, suppose that M is the RREF of A. Then $A\mathbf{x} = \mathbf{0}$ and $M\mathbf{x} = \mathbf{0}$ have the same solutions. The solutions of $M\mathbf{x} = \mathbf{0}$ are easy to read off.

- To elaborate a bit on the previous point: Let M be the RREF of A, which is obtained from A by multiplying on the left by elementary matrices: $E_k \cdots E_1 A = M$. Using this fact, show that $A\mathbf{x} = \mathbf{0}$ if and only if $M\mathbf{x} = \mathbf{0}$. (It follows from this that $\mathcal{R}(A) = \mathcal{N}(A)^\perp = \mathcal{N}(M)^\perp = \mathcal{R}(M)$, which is why the nonzero rows of M give a basis for $\mathcal{R}(A)$.)

- Know criteria for the existence of inverses. Given an $m \times n$ matrix A:

- A has a left inverse if and only if $\text{rank}(A) = n$.
- A has a right inverse if and only if $\text{rank}(A) = m$.
- If A is not square then left (right) inverses are not unique.
- A^{-1} exists if and only if $m = n = \text{rank}(A)$.
- Two-sided inverses are unique.

Be able to compute one-sided and two-sided inverses for small matrices.

- A linear system $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b} \in \mathcal{C}(A)$, in which case the solution has the form
$$\mathbf{x}' + \mathcal{N}(A) = \{\mathbf{x}' + \mathbf{x} : \mathbf{x} \in \mathcal{N}(A)\}.$$
If A has independent columns then $\mathcal{N}(A) = \{\mathbf{0}\}$ so the solution is a single point.

- Compute the solution of a small linear system.

- Use the trick $\mathbf{x}^T A^T A\mathbf{x} = \|A\mathbf{x}\|^2$ to prove that $\mathcal{N}(A^T A) = \mathcal{N}(A)$. One direction: If $A\mathbf{x} = \mathbf{0}$ then $A^T A\mathbf{x} = A^T\mathbf{0} = \mathbf{0}$. Other direction: If $A^T A\mathbf{x} = \mathbf{0}$ then $\|A\mathbf{x}\|^2 = \mathbf{x}^T A^T A\mathbf{x} = \mathbf{x}^T\mathbf{0} = \mathbf{0}$. But $\|A\mathbf{x}\|^2 = 0$ implies $A\mathbf{x} = 0$ by properties of norms.

- For any matrix A, the matrix $A^T A$ is square and symmetric.

- If A has independent columns, show that $A^T A$ also has independent columns, hence $(A^T A)^{-1}$ exists. Do the same for AA^T when A has independent rows.

- If $A\mathbf{x} = \mathbf{b}$ has no solution, multiply both sides on the left by $A^T A\mathbf{x} = A^T\mathbf{b}$. The new system always has solutions, and these solutions minimize $\|A\mathbf{x} - \mathbf{b}\|$. If A has independent columns then the least-squares solution is unique:
$$A^T A\mathbf{x} = A^T\mathbf{b}$$
$$\mathbf{x} = (A^T A)^{-1} A^T\mathbf{b}.$$

- Solve a small least squares problem, such as fitting a line to three data points, or finding the distance between skew lines in $\mathbb{R}^3$.

- **Projection.** Let $P\mathbf{x}$ be the projection of $\mathbf{x}$ onto $\mathcal{C}(A)$. Since $P\mathbf{x}$ is in $\mathcal{C}(A)$ we must have $P\mathbf{x} = A\hat{\mathbf{x}}$ for some $\hat{\mathbf{x}}$. We also know that $P\mathbf{x} - \mathbf{x}$ is orthogonal to $\mathcal{C}(A)$, which means that $P\mathbf{x} - \mathbf{x}$ is orthogonal to every column of A:
$$A^T(P\mathbf{x} - \mathbf{x}) = \mathbf{0}.$$

- Assuming A has independent colums, solve the previous equation to get
$$P = A(A^T A)^{-1} A^T.$$

- In general, P is a projection when $P^2 = P$ and $P^T = P$. If P is a projection show that $Q = I - P$ is also a projection. In fact, P and Q project onto orthogonal subspaces. This sometimes gives a shortcut to compute P. For example, let P be the projection onto the plane $ax + by + cz = 0$. Then $I - P$ projects onto the line generated by (a, b, c):

$$\begin{aligned} I - P &= \begin{pmatrix} a \\ b \\ c \end{pmatrix} \left[\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right]^{-1} \begin{pmatrix} a & b & c \end{pmatrix} \\ &= \frac{1}{a^2 + b^2 + c^2} \begin{pmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{pmatrix}. \end{aligned}$$

- State the definition of k-linear forms. Know that every 1-linear form looks like $\varphi_{\mathbf{b}}(\mathbf{x}) = \mathbf{b}^T\mathbf{x}$ for a vector $\mathbf{b}$. Know that every 2-linear form looks like $\varphi_B(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T B\mathbf{y}$ for a square matrix B.

- Relate properties of the function φ_B to properties of the matrix B:

  - $B = C$ if and only if $\varphi_B(\mathbf{x}, \mathbf{y}) = \varphi_C(\mathbf{x}, \mathbf{y})$ for all $\mathbf{x}, \mathbf{y}$.
  - $B^T = B$ if and only if $\varphi_B(\mathbf{x}, \mathbf{y}) = \varphi_B(\mathbf{y}, \mathbf{x})$ for all $\mathbf{x}, \mathbf{y}$.
  - If $B = A^T A$ then $\varphi_B(\mathbf{x}, \mathbf{x}) \geq 0$ for all $\mathbf{x}$.
  - If $B = A^T A$ and A has independent columns then $\varphi_B(\mathbf{x}, \mathbf{x}) = 0$ implies $\mathbf{x} = \mathbf{0}$.

- Write a given polynomial $f(\mathbf{x})$ of degree 2 in the form $f(\mathbf{x}) = b + \mathbf{b}^T\mathbf{x} + \mathbf{x}^T B\mathbf{x}$ for some scalar b, vector $\mathbf{b}$ and symmetric matrix B.

- Use Laplace expansion or some other method to compute small determinants.

- Know that A^{-1} exists if and only if $\det(A) \neq 0$.

- Know the formulas

  - $\det(A^T) = \det(A)$
  - $\det(AB) = \det(A)\det(B)$
  - $\det(A^{-1}) = 1/\det(A)$.

- If A is square, prove that $\sqrt{\det(A^T A)} = |\det(A)|$.

- If A is $n \times k$, know that $\sqrt{\det(A^T A)}$ is the k-volume of the k-parallelogram in $\mathbb{R}^n$ generated by the columns of A.