MTH 505: Number Theory **Spring 2020**
Homework 5 **Drew Armstrong**

Problem 1. $\mathbb{Z}[\sqrt{-1}]$ is Euclidean. Consider the ring of Gaussian integers:

$$\mathbb{Z}[\sqrt{-1}] = \{a + b\sqrt{-1} : a, b \in \mathbb{Z}\}.$$

For any γ, δ we observe that $\sqrt{N(\gamma - \delta)}$ is the distance between γ and δ in the complex plane. For all $\alpha, \beta \in \mathbb{Z}[\sqrt{-1}]$ with $\beta \neq 0$, use this geometric interpretation to prove that there exist some (possibly non-unique) $\chi, \rho \in \mathbb{Z}[\sqrt{-1}]$ such that

$$\begin{cases} \alpha = \chi\beta + \rho, \\ N(\rho) < N(\beta). \end{cases}$$

[Hint: The set of numbers $\{\chi\beta : \chi \in \mathbb{Z}[\sqrt{-1}]\}$ forms a square grid in the complex plane with side length $\sqrt{N(\beta)}$. Let $\chi\beta$ be the (possibly non-unique) grid point closest to α and define $\rho := \alpha - \chi\beta$. Draw a picture to show that $\sqrt{N(\rho)} < \sqrt{N(\beta)}$.]

Problem 2. $\mathbb{Z}[\sqrt{-5}]$ is not Euclidean. Consider the ring

$$\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\}.$$

(a) Prove that 2 is irreducible in $\mathbb{Z}[\sqrt{-5}]$. [Hint: If $2 = \alpha\beta$ for some non-units $\alpha, \beta \in \mathbb{Z}[\sqrt{-5}]$ then we must have $N(\alpha) = N(\beta) = 2$. But show that there **do not exist** any elements of norm 2 in the ring $\mathbb{Z}[\sqrt{-5}]$.]

(b) Observe that $2 \cdot 3 = 6 = (1 + \sqrt{-5})(1 - \sqrt{-5})$ and hence 2 divides the product $(1 + \sqrt{-5})(1 - \sqrt{-5})$. But show that 2 does not divide $1 + \sqrt{-5}$ or $1 - \sqrt{-5}$. [Hint: Suppose that $2(a + b\sqrt{-5}) = 1 + \sqrt{-5}$ for some integers $a, b \in \mathbb{Z}$.]

Problem 3. Pell's Equation. Use the method of continued fractions to find the complete integer solution $x, y \in \mathbb{Z}$ (with $x, y \geqslant 0$) to the equations

$$x^2 - dy^2 = +1 \quad \text{and} \quad x^2 - dy^2 = -1$$

in the following two cases:

(a) $d = 13$

(b) $d = 23$

[Remark: In one of these cases you will find that the equation $x^2 - dy^2 = -1$ has no solution.]