Math 461 **Spring 2023**
Homework 4 **Drew Armstrong**

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1. Conjugation of Complex Polynomials. For any polynomial $f(x) = \sum \alpha_k x^k \in \mathbb{C}[x]$ we define the *conjugate polynomial* by taking the complex conjugate of each coefficient:

$$f^*(x) := \sum \alpha_k^* x^k \in \mathbb{C}[x].$$

(a) For any complex polynomial $f(x) \in \mathbb{C}[x]$ and complex number $\alpha \in \mathbb{C}$, show that

$$[f(\alpha)]^* = f^*(\alpha^*).$$

(b) For any complex polynomial $f(x) \in \mathbb{C}[x]$ show that

$$f(x) = f^*(x) \quad \Longleftrightarrow \quad f(x) \in \mathbb{R}[x].$$

(c) For any complex polynomials $f(x), g(x) \in \mathbb{C}[x]$, show that

$$(f+g)^*(x) = f^*(x) + g^*(x) \quad \text{and} \quad (fg)^*(x) = f^*(x)g^*(x).$$

(d) For any complex polynomial $f(x) \in \mathbb{C}[x]$, show that

$$f(x) + f^*(x) \in \mathbb{R}[x] \quad \text{and} \quad f(x)f^*(x) \in \mathbb{R}[x].$$

2. Invariance of Quotient and Remainder. Consider two fields $\mathbb{F} \subseteq \mathbb{E}$, so we can think of $\mathbb{F}[x]$ is a subring of $\mathbb{E}[x]$.

(a) Given $f(x), g(x) \in \mathbb{F}[x]$ such that $g(x) \neq 0$, suppose we have polynomials $q(x), r(x)$ with coefficients in $\mathbb{E}$ satisfying

$$\begin{cases} f(x) = q(x)g(x) + r(x), \\ r(x) = 0 \text{ or } \deg(r) < \deg(g). \end{cases}$$

In this case, prove that $q(x)$ and $r(x)$ actually have coefficients in $\mathbb{F}$. [Hint: Divide $f(x)$ by $g(x)$ in the ring $\mathbb{F}[x]$ to obtain a quotient and remainder $q'(x), r'(x) \in \mathbb{F}[x]$. Now use the **uniqueness** of quotient and remainder in the ring $\mathbb{E}[x]$ (HW1.3).]

(b) Now consider the field extension $\mathbb{C} \supseteq \mathbb{R}$. Suppose that the real polynomial $f(x) \in \mathbb{R}[x]$ has a complex root $\alpha \in \mathbb{C}$ that is not real (i.e., $\alpha \neq \alpha^*$). In this case, prove that there exists a **real** polynomial $h(x) \in \mathbb{R}[x]$ satisfying

$$f(x) = (x-\alpha)(x-\alpha^*)h(x).$$

3. Equivalent Forms of the FTA.

(a) Suppose that every (non-constant) real polynomial $f(x) \in \mathbb{R}[x]$ has a root in $\mathbb{C}$. Use Problem 2 to prove that every (non-constant) real polynomial factors as a product of real polynomials of degrees 1 and 2.

(b) Suppose that every (non-constant) real polynomial $f(x) \in \mathbb{R}[x]$ has a root in $\mathbb{C}$. In this case, use Problem 1 to prove that every (non-constant) complex polynomial $f(x) \in \mathbb{C}[x]$ has a root in $\mathbb{C}$. [Hint: Let $f(x) \in \mathbb{C}[x]$ be any (non-constant) complex polynomial and consider the polynomial $g(x) = f(x)f^*(x)$.]

4. Biquadratic Polynomials. Given real numbers $a, b \in \mathbb{R}$ we will prove that there exist real numbers $p, q, r, s \in \mathbb{R}$ satisfying

$$x^4 + ax^2 + b = (x^2 + px + r)(x^2 + qx + s).$$

(a) If $a^2 - 4b \geq 0$, show that $x^4 + ax^2 + b = (x^2 - A)(x^2 - B)$ for some real $A, B \in \mathbb{R}$.

(b) If $a^2 - 4b < 0$, show that $x^4 + ax^2 + b = (x^2 - \alpha)(x^2 - \alpha^*)$ for some non-real $\alpha \in \mathbb{C}$.

(c) Continuing from (b), let $\pm\beta \in \mathbb{C}$ be the roots of $x^2 - \alpha \in \mathbb{C}[x]$ and let $\pm\gamma \in \mathbb{C}$ be the roots of $x^2 - \alpha^* \in \mathbb{C}[x]$, so that

$$x^4 + ax^2 + b = (x - \beta)(x + \beta)(x - \gamma)(x + \gamma).$$

Show that $\beta^* = \gamma$ or $\beta^* = -\gamma$. Now what?

5. Laplace's Proof of the FTA. Consider any field extension $\mathbb{E} \supseteq \mathbb{R}$ and let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{E}$ be any elements satisfying

$$x^3 + x + 1 = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3).$$

(a) Expand the right hand side and compare coefficients to find formulas for

$$\alpha_1 + \alpha_2 + \alpha_3, \quad \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3, \quad \text{and} \quad \alpha_1\alpha_2\alpha_3.$$

(b) Now let $t \in \mathbb{R}$ be any real number and consider the polynomial

$$g_t(x) = (x - \beta_{12t})(x - \beta_{13t})(x - \beta_{23t}),$$

where $\beta_{ijt} = \alpha_i + \alpha_j + t\alpha_i\alpha_j$ for all pairs $1 \leq i < j \leq 3$. Find explicit formulas for the coefficients of $g_t(x)$ in terms of t, and conclude that $g_t(x)$ has **real coefficients**. [Hint: Each coefficient g_t is a symmetric function of $\alpha_1, \alpha_2, \alpha_3$. Any symmetric function of $\alpha_1, \alpha_2, \alpha_3$ can be expressed in terms of the *elementary symmetric functions* of $\alpha_1, \alpha_2, \alpha_3$, as listed in part (a).]