Math 461 **Spring 2023**
Homework 3 **Drew Armstrong**

1. Complex Conjugation. For any complex number $a + ib \in \mathbb{C}$ we define its *complex conjugate* $\alpha^* := a - ib \in \mathbb{C}$.

(a) For any $\alpha \in \mathbb{C}$, show that $\alpha = \alpha^*$ if and only if $\alpha \in \mathbb{R}$.
(b) For any $\alpha, \beta \in \mathbb{C}$ show that $(\alpha + \beta)^* = \alpha^* + \beta^*$ and $(\alpha\beta)^* = \alpha^*\beta^*$.
(c) For any real polynomial $f(x) \in \mathbb{R}[x]$ and complex number $\alpha \in \mathbb{C}$, combine parts (a) and (b) to show that $f(\alpha)^* = f(\alpha^*)$.
(d) For any complex number $\alpha \in \mathbb{C}$ show that the polynomial $(x - \alpha)(x - \alpha^*)$ has real coefficients. [Hint: Show that $\alpha + \alpha^*$ and $\alpha\alpha^*$ are real.]

(a): For any complex number $\alpha = a + ib \in \mathbb{C}$, note that

$$\begin{aligned} \alpha = \alpha^* \quad &\Longleftrightarrow \quad a + ib = a - ib \\ &\Longleftrightarrow \quad b = -b \\ &\Longleftrightarrow \quad 2b = 0 \\ &\Longleftrightarrow \quad b = 0 \\ &\Longleftrightarrow \quad \alpha \in \mathbb{R}. \end{aligned}$$

(b): Let $\alpha = a + bi$ and $\beta = c + di$. Then we have

$$\begin{aligned} (\alpha + \beta)^* &= (a + bi + c + di)^* \\ &= [(a + c) + (b + d)i]^* \\ &= (a + c) - (b + d)i \\ &= (a - bi) + (c - di) \\ &= \alpha^* + \beta^* \end{aligned}$$

and

$$\begin{aligned} \alpha^*\beta^* &= (a + ib)^*(c + id)^* \\ &= (a - ib)(c - id) \\ &= (ac - bd) + (-ad - bc)i \\ &= (ac - bd) - (ad + bc)i \\ &= [(ac - bd) + (ad + bc)i]^* \\ &= (\alpha\beta)^*. \end{aligned}$$

This second identity becomes less mysterious if we use the polar form. Let $\alpha = re^{i\theta}$ and $\beta = se^{i\mu}$.[1] From Euler's formula we know that $(re^{i\theta})^* = re^{-i\theta}$. Hence

$$\begin{aligned} \alpha^*\beta^* &= (re^{i\theta})^*(se^{i\mu})^* \\ &= re^{-i\theta}se^{-i\mu} \\ &= (rs)e^{-i(\theta+\mu)} \\ &= (rse^{i(\theta+\mu)})^* \\ &= (\alpha\beta)^*. \end{aligned}$$

[1] By using the word "polar form" we implicitly assume that $r, s, \theta, \mu \in \mathbb{R}$ and $r, s \geq 0$.

(c): Consider a polynomial $f(x) = \sum a_k x^k$ with real coefficients $a_k \in \mathbb{R}$, and let $\alpha \in \mathbb{C}$ be any complex number. Then combining parts (a) and (b) gives

$$\begin{aligned}
f(\alpha)^* &= \left(\sum a_k x^k\right)^* && \\
&= \sum (a_k \alpha^k)^* && * \text{ preserves addition} \\
&= \sum a_k^* (\alpha^*)^k && * \text{ preserves multiplication} \\
&= \sum a_k (\alpha^*)^k && a_k \text{ is real} \\
&= f(\alpha^*).
\end{aligned}$$

(d): Consider a complex number $\alpha = a + ib$. We observe that $\alpha + \alpha^*$ and $\alpha\alpha^*$ are real numbers:

$$\begin{aligned}
\alpha + \alpha^* &= (a + ib) + (a - ib) = 2a + 0i, \\
\alpha\alpha^* &= (a + ib)(a - ib) = (a^2 + b^2) + 0i.
\end{aligned}$$

It follows that the polynomial $(x - \alpha)(x - \alpha^*)$ has real coefficients:

$$\begin{aligned}
(x - \alpha)(x - \alpha^*) &= x^2 - (\alpha + \alpha^*)x + \alpha\alpha^2 \\
&= x^2 - 2ax + (a^2 + b^2).
\end{aligned}$$

Another Proof: We can also use parts (a) and (b) to show that $\alpha + \alpha^*$ and $\alpha\alpha^*$ are real. To do this, we use part (b) to show that

$$(\alpha + \alpha^*)^* = \alpha^* + (\alpha^*)^* = \alpha^* + \alpha = \alpha + \alpha^*$$

and

$$(\alpha\alpha^*)^* = \alpha^*(\alpha^*)^* = \alpha^*\alpha = \alpha\alpha^*.$$

But from part (a) any complex number that is equal to its own conjugate is real. This proof is nice because we don't need to waste letters of the alphabet. It also generalizes to fancier situations.

2. Roots of Unity. Recall Euler's formula

$$e^{it} = \cos t + i \sin t \quad \text{for any real number } t \in \mathbb{R}.$$

Fix an integer $n \geq 1$ and let $\omega = e^{i2\pi/n}$.

(a) For any integer $k \in \mathbb{Z}$, use Euler's formula to show that $(\omega^k)^n = 1$.

(b) For any integers $k, \ell \in \mathbb{Z}$, use Euler's formula to show that

$$\omega^k = \omega^\ell \quad \Longleftrightarrow \quad k - \ell = mn \text{ for some integer } m \in \mathbb{Z}.$$

(c) For any integer $k \in \mathbb{Z}$, use Euler's formula to show that $(\omega^k)^* = \omega^{-k}$.

(d) It follows from (a) and (b) that the polynomial $x^n - 1$ can be factored as

$$x^n - 1 = (x - \omega^0)(x - \omega^1)(x - \omega^2) \cdots (x - \omega^{n-1}).$$

Use this factorization to show that

$$x^n - \alpha^n = (x - \omega^0\alpha)(x - \omega^1\alpha)(x - \omega^2\alpha) \cdots (x - \omega^{n-1}\alpha)$$

for any complex number $\alpha \in \mathbb{C}$. [Hint: Replace x by x/α.]

(a): For any integer $k \in \mathbb{Z}$ we have

$$\begin{aligned}(\omega^k)^n &= \omega^{kn} \\ &= (e^{i2\pi/n})^{kn} \\ &= e^{i2\pi k} \\ &= \cos(2\pi k) + i\sin(2\pi k) \\ &= 1 + 0i \\ &= 1.\end{aligned}$$

Alternatively, first observe that

$$\omega^n = (e^{i2\pi/n})^n = e^{i2\pi} = \cos(2\pi) + i\sin(2\pi) = 1 + 0i = 1,$$

and then just use algebra:

$$(\omega^k)^n = \omega^{kn} = (\omega^n)^k = 1^k = 1.$$

It is always better when a proof follows from "just algebra". Then you don't have to think.

(b): First we observe that $e^{i\theta} = 1$ if and only if $\theta = 2\pi m$ for some integer $m \in \mathbb{Z}$. Indeed,

$$\begin{aligned}e^{i\theta} = 1 \quad &\Longleftrightarrow \quad \cos\theta + i\sin\theta = 1 + 0i \\ &\Longleftrightarrow \quad \cos\theta = 1 \text{ and } \sin\theta = 0 \\ &\Longleftrightarrow \quad \theta = 2\pi m \text{ for some } m \in \mathbb{Z}.\end{aligned}$$

The final equivalence is just a statement about the unit circle. The point on the unit circle with coordinates $(1, 0)$ makes angle 0 with the x-axis. And the angle 0 is equivalent to $2\pi m$ for any $m \in \mathbb{Z}$.

Now fix $n \geq 1$ and $\omega = e^{i2\pi/n}$. Then for any integers $k, \ell \in \mathbb{Z}$ we have

$$\begin{aligned}\omega^k = \omega^\ell \quad &\Longleftrightarrow \quad \omega^{k-\ell} = 1 \\ &\Longleftrightarrow \quad e^{i2\pi(k-\ell)/n} = 1 \\ &\Longleftrightarrow \quad 2\pi(k-\ell)/n = 2\pi m \text{ for some } m \in \mathbb{Z} \\ &\Longleftrightarrow \quad k - \ell = mn \text{ for some } m \in \mathbb{Z}.\end{aligned}$$

(c): Fix $\omega = e^{i2\pi/n}$. Then for any integer $k \in \mathbb{Z}$ we have

$$\begin{aligned}\omega^{-k} &= (e^{i2\pi/n})^{-k} \\ &= e^{i(-2\pi k/n)} \\ &= \cos\left(-\frac{2\pi k}{n}\right) + i\sin\left(-\frac{2\pi k}{n}\right) \\ &= \cos\left(\frac{2\pi k}{n}\right) - i\sin\left(\frac{2\pi k}{n}\right).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}(\omega^k)^* &= (e^{i2\pi k/n})^* \\ &= \left[\cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right)\right]^* \\ &= \cos\left(\frac{2\pi k}{n}\right) - i\sin\left(\frac{2\pi k}{n}\right).\end{aligned}$$

Remark: It follows from this that

$$\begin{aligned}(x-\omega^k)(x-\omega^{-k}) &= x^2-(\omega^k+\omega^{-k})x+\omega^k\omega^{-k}\\ &= x^2-2\cos(2\pi k/n)x+1.\end{aligned}$$

(d): Fix $\omega = e^{i2\pi/n}$. From part (b) we know that the n complex numbers

$$\omega^0, \omega^1, \ldots, \omega^{n-1}$$

are distinct. Geometrically, these are n equally spaced points around the unit circle in the complex plane. And we know from (a) that each of these numbers is a root of the polynomial $x^n - 1$. Hence from Descartes' Theorem we have

$$x^n - 1 = (x-\omega^0)(x-\omega^1)\cdots(x-\omega^{n-1}).$$

Now consider a nonzero complex number $\alpha \neq 0$ and replace x by x/α to obtain

$$\left(\frac{x}{\alpha}\right)^n - 1 = \left(\frac{x}{\alpha}-\omega^0\right)\left(\frac{x}{\alpha}-\omega^1\right)\cdots\left(\frac{x}{\alpha}-\omega^{n-1}\right).$$

Then multiplying both sides by α^n gives

$$\begin{aligned}\alpha^n\left[\left(\frac{x}{\alpha}\right)^n-1\right] &= \alpha^n\left(\frac{x}{\alpha}-\omega^0\right)\left(\frac{x}{\alpha}-\omega^1\right)\cdots\left(\frac{x}{\alpha}-\omega^{n-1}\right)\\ \alpha^n\left[\frac{x^n}{\alpha^n}-1\right] &= \alpha\left(\frac{x}{\alpha}-\omega^0\right)\alpha\left(\frac{x}{\alpha}-\omega^1\right)\cdots\alpha\left(\frac{x}{\alpha}-\omega^{n-1}\right)\\ x^n-\alpha^n &= (x-\omega^0\alpha)(x-\omega^1\alpha)(x-\omega^2\alpha)\cdots(x-\omega^{n-1}\alpha).\end{aligned}$$

We have proved this formula for $\alpha \neq 0$, but we observe that it also works for $\alpha = 0$.

3. Leibniz' Mistake. Fix a positive real number $a > 0$. In 1702, Leibniz claimed that the polynomial $x^4 + a^4$ cannot be factored over the real numbers. In this problem you will show that Leibniz was wrong.

(a) Let $\lambda = e^{i\pi/4} = (1+i)/\sqrt{2}$. Use Euler's formula to show that $\lambda^2 = i$ and $\lambda^4 = -1$.
(b) Substitute $\alpha = \lambda a$ into Problem 2(d) and use the idea from Problem 1(d) to show that

$$x^4 + a^4 = (x^2 + a\sqrt{2}x + a^2)(x^2 - a\sqrt{2}x + a^2).$$

[It's easy to **check** that this factorization it correct. I want you to **derive** the factorization using properties of complex numbers.]

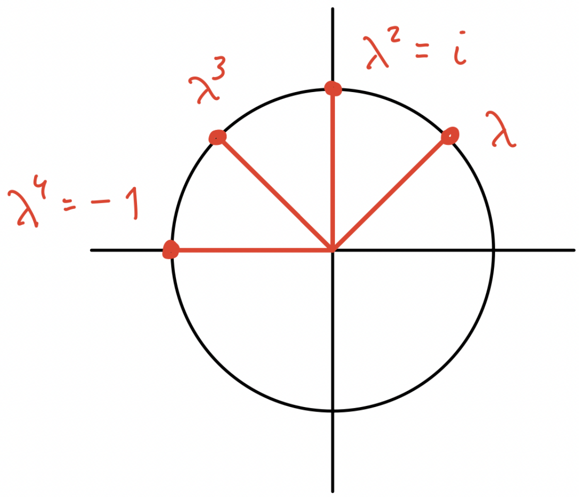
(a): If $\lambda = e^{i\pi/4}$ then

$$\lambda^2 = (e^{i\pi/4})^2 = e^{i\pi/2} = \cos(\pi/2) + i\sin(\pi/2) = 0 + i = i$$

and

$$\lambda^4 = (e^{i\pi/4})^4 = e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1 + 0i = -1.$$

Picture:



Remark: What are the 4th roots of -1? If λ is a 4th of -1 then we must have

$$\begin{aligned} \lambda^4 &= -1 \\ |\lambda^4| &= |-1| \\ |\lambda|^4 &= 1, \end{aligned}$$

and since $|\lambda|$ is a non-negative real number we must have $|\lambda| = 1$. This implies that $\lambda = e^{i\theta}$ for some angle θ. But then since $-1 = e^{i\theta}$ we must have

$$\begin{aligned} \lambda^4 &= -1 \\ (e^{i\theta})^4 &= e^{i\pi} \\ e^{i4\theta} &= e^{i\pi}, \end{aligned}$$

which implies that $4\theta - \pi = 2\pi m$ for some integer $m \in \mathbb{Z}$. In other words:

$$\begin{aligned} \theta &= \frac{\pi}{4} + \frac{\pi}{2}m \quad \text{for some integer } m \in \mathbb{Z}. \\ &= \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}. \end{aligned}$$

It follows that the 4th roots of -1 are

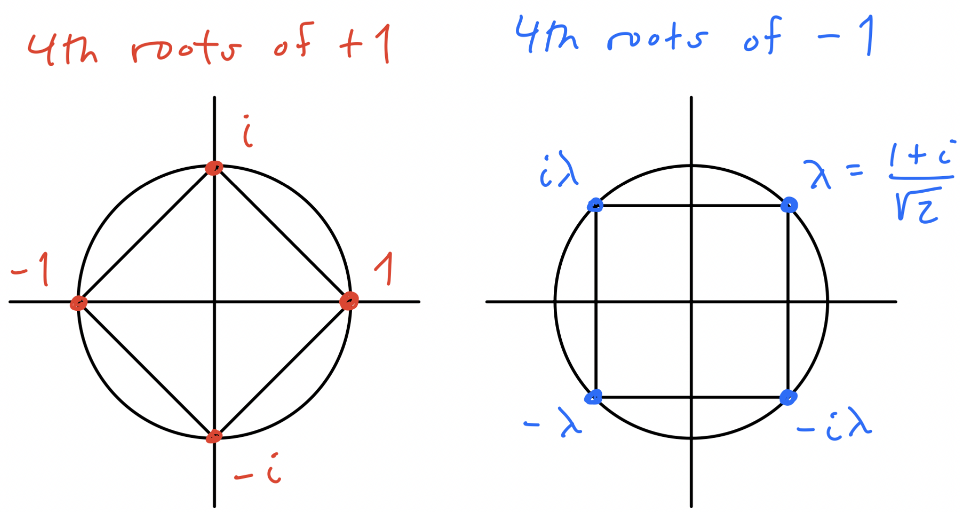
$$\begin{aligned} e^{i\pi/4} &= (1+i)/\sqrt{2}, \\ e^{i3\pi/4} &= (-1+i)/\sqrt{2}, \\ e^{i5\pi/4} &= (-1-i)/\sqrt{2}, \\ e^{i7\pi/4} &= (1-i)/\sqrt{2}. \end{aligned}$$

Another point of view: If we let $\lambda = e^{i\pi/4} = (1+i)/\sqrt{2}$ denote the **principal 4th root of** -1 and let $\omega = e^{i2\pi/4} = i$ denote the **principal 4th root of 1,** then the 4th roots of 1 are

$1, \omega, \omega^2, \omega^3 = 1, i, -1, -i$ and the 4th roots of -1 are

$$\begin{aligned}\lambda, \lambda\omega, \lambda\omega^2, \lambda\omega^3 &= \lambda, i\lambda, -\lambda, -i\lambda \\ &= (1+i)/\sqrt{2}, i(1+i)/\sqrt{2}, -(1+i)/\sqrt{2}, -i(1+i)/\sqrt{2} \\ &= (1+i)/\sqrt{2}, (-1+i)/\sqrt{2}, (-1-i)/\sqrt{2}, (1-i)/\sqrt{2},\end{aligned}$$

as we have just seen. Picture:



(b): Let a be real and let $\lambda = e^{i\pi/4} = (1+i)/\sqrt{2}$, so that

$$(\lambda a)^4 = \lambda^4 a^4 = -a^4.$$

Thus λa is a 4th root of $-a^4$. Substituting $\alpha = \lambda a$ into 2(d) gives

$$\begin{aligned}x^4 - \alpha^4 &= (x-\alpha)(x-i\alpha)(x+\alpha)(x+i\alpha) \\ x^4 + a^4 &= (x-\lambda a)(x-i\lambda a)(x+\lambda a)(x+i\lambda a).\end{aligned}$$

Since a is real, we observe that the numbers $\lambda a = (a+ai)/\sqrt{2}$ and $-i\lambda a = (a-ai)/\sqrt{2}$ are conjugate, while the numbers $-\lambda a = (-a-ai)/\sqrt{2}$ and $i\lambda a = (-a+ai)/\sqrt{2}$ are conjugate. It follows from 1(d) that the polynomials $(x-\lambda a)(x+i\lambda a)$ and $(x+\lambda a)(x-i\lambda a)$ have real coefficients. To be specific, we can check that[2]

$$(x-\lambda a)(x+i\lambda a) = x^2 - a\sqrt{2}x + a^2$$

and

$$(x+\lambda a)(x-i\lambda a) = x^2 + a\sqrt{2}x + a^2,$$

hence

$$x^4 + a^4 = (x^2 + a\sqrt{2}x + a^2)(x^2 - a\sqrt{2}x + a^2).$$

Remark: This factorization guarantees that the integral of $1/(x^4+a^4)$ can be expressed in terms of exponential and trigonometric functions, which Leibniz claimed is impossible.

[2]The sum of conjugates is just 2 times the real part. The quick way to verify that $(\lambda a)(-i\lambda a) = a^2$ is to use the fact that $\lambda^2 = i$:

$$(\lambda a)(-i\lambda a) = -i\lambda^2 a^2 = -i(i)a^2 = +a^2.$$

More generally, the Fundamental Theorem of Algebra guarantees that any real polynomial $f(x) \in \mathbb{R}[x]$ can be factored as a product of real polynomials of degrees 1 and 2.

4. Fifth Roots of Unity. Let $\omega = e^{i2\pi/5}$ so that

$$x^5 - 1 = (x - \omega^0)(x - \omega^1)(x - \omega^2)(x - \omega^3)(x - \omega^4).$$

(a) Use this factorization to show that $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$. [Hint: Expand the right hand side and compare coefficients.]
(b) Show that $\omega^3 = \omega^{-2}$ and $\omega^4 = \omega^{-1}$, so that $1 + \omega + \omega^2 + \omega^{-2} + \omega^{-1} = 0$. [Hint: 2(b)]
(c) Let $\alpha = \omega + \omega^{-1}$ and use part (b) to show that $\alpha^2 + \alpha - 1 = 0$.
(d) Solve the quadratic equation in (c) to get an explicit formula for $\cos(2\pi/5)$. [Hint: We know from Euler's formula or Problem 2(c) that $\omega + \omega^{-1} = 2\cos(2\pi/5)$.]

(a): Expanding the right hand side gives

$$\begin{aligned} x^5 - 1 &= (x-1)(x-\omega^1)(x-\omega^2)(x-\omega^3)(x-\omega^4) \\ &= x^5 \\ &\quad - (1 + \omega + \omega^2 + \omega^3 + \omega^4)x^4 \\ &\quad + (\text{sum of products of pairs of the roots})x^3 \\ &\quad - (\text{sum of products of triples of the roots})x^2 \\ &\quad + (\text{sum of products of quadruples of the roots})x \\ &\quad - 1 \cdot \omega \cdot \omega^2 \cdot \omega^3 \cdot \omega^4. \end{aligned}$$

Since the coefficient of x^4 in the left side is zero, we conclude that

$$1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0.$$

Remark: Looking at the other coefficients gives other interesting identities; for example:

$$1 \cdot \omega \cdot \omega^2 \cdot \omega^3 \cdot + 1 \cdot \omega \cdot \omega^2 \cdot \cdot \omega^4 + 1 \cdot \omega \cdot \cdot \omega^3 \cdot \omega^4 + 1 \cdot \cdot \omega^2 \cdot \omega^3 \cdot \omega^4 + \omega \cdot \omega^2 \cdot \omega^3 \cdot \omega^4 = 0.$$

Another Proof: We know that

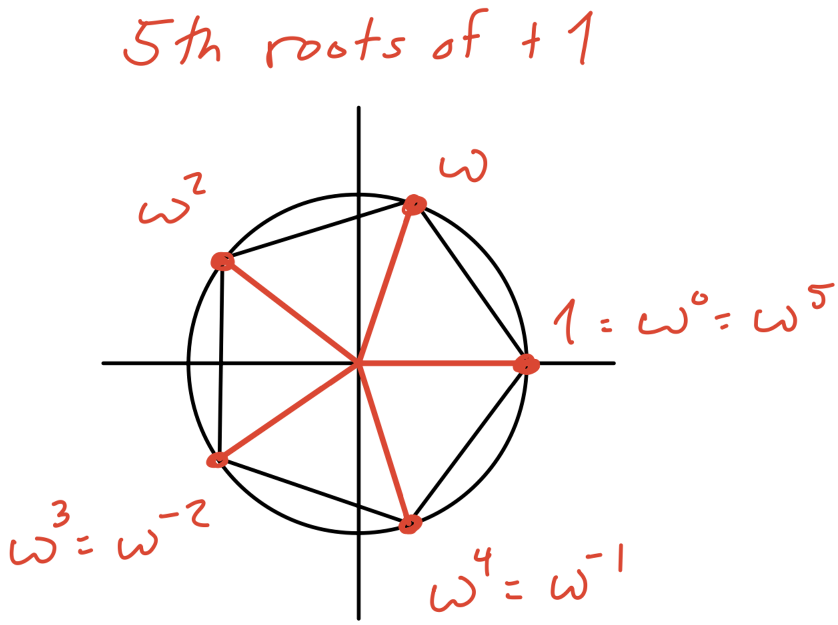
$$x^5 - 1 = (x-1)(1 + x + x^2 + x^3 + x^4).$$

We also know that $\omega^5 - 1 = 0$ and $\omega - 1 \neq 0$, so substituting $x = \omega$ into the previous equation gives the desired result.

Yet Another Proof: Geometrically, the numbers $1, \omega, \omega^2, \omega^3, \omega^4$ are the vertices of a regular pentagon in the complex plane, centered at the origin. But addition in the complex plane is the same as vector addition, so we obtain

$$\begin{aligned} 0 &= \text{center of mass of the five points} \\ &= (1 + \omega + \omega^2 + \omega^3 + \omega^4)/5. \end{aligned}$$

Picture:



(b): If $\omega^{i2\pi/n}$ then we know from 2(b) that

$$\omega^k = \omega^\ell \quad \Longleftrightarrow \quad k - \ell = mn \text{ for some integer } m \in \mathbb{Z}.$$

In our case we have $n = 5$, so that

$$\omega^k = \omega^\ell \quad \Longleftrightarrow \quad k - \ell = 5m \text{ for some integer } m \in \mathbb{Z}.$$

It follows that $\omega^3 = \omega^{-2}$ and $\omega^4 = \omega^{-1}$.

(c): Define $\alpha = \omega + \omega^{-1}$. Then from parts (a) and (b) we have

$$\begin{aligned}
\alpha^2 + \alpha - 1 &= (\omega + \omega^{-1})^2 + (\omega + \omega^{-1}) - 1 \\
&= (\omega^2 + 2 + \omega^{-2}) + (\omega + \omega^{-1}) - 1 \\
&= 1 + \omega + \omega^2 + \omega^{-2} + \omega^{-1} \\
&= 0.
\end{aligned}$$

(d): We also know from 2(c) that

$$\begin{aligned}
\alpha &= \omega + \omega^{-1} \\
&= \left[\cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right)\right] + \left[\cos\left(\frac{2\pi}{5}\right) - i\sin\left(\frac{2\pi}{5}\right)\right] \\
&= 2\cos\left(\frac{2\pi}{5}\right).
\end{aligned}$$

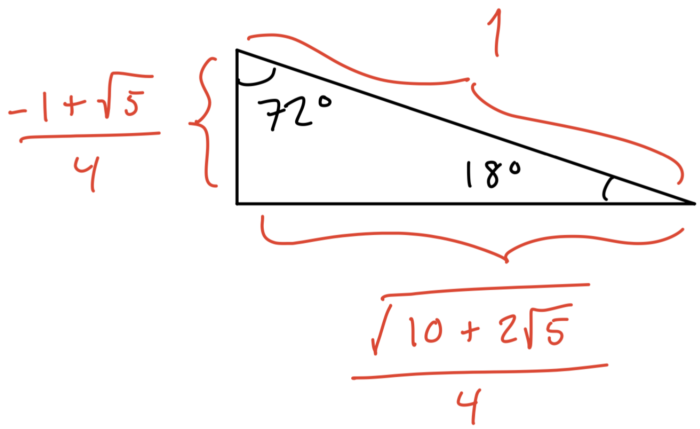
Hence the quadratic formula gives

$$2\cos\left(\frac{2\pi}{5}\right) = \alpha = \frac{-1 \pm \sqrt{1^2 - 4(-1)}}{2} = \frac{-1 \pm \sqrt{5}}{2}.$$

Finally, since $2\pi/5$ is less than $90°$ we know that $\cos(2\pi/5) > 0$, hence we must have

$$\cos\left(\frac{2\pi}{5}\right) = \frac{-1+\sqrt{5}}{4}.$$

You can verify with a calculator that this is correct. Thus, to our favorite $30°, 60°, 90°$ and $45°, 45°, 90°$ triangles, we can add the $18°, 72°, 90°$ triangle:



Remark: Why is this identity never taught? I don't know. It's only a bit more complicated than the identities for $\cos(2\pi/n)$ when $n = 3, 4, 6$. I guess it's a bit less useful.

Remark: Later we will discuss the *Gauss-Wantzel Theorem*, which says that the number $\cos(2\pi/n)$ can be expressed in terms of integers and square roots if and only if $n = 2^k p_1 p_2 \cdots p_\ell$, where $p_1, \ldots, p_\ell$ are distinct Fermat primes. In particular, since 7 and 9 are not of this form, the numbers $\cos(2\pi/7)$ and $\cos(2\pi/9)$ **cannot** be expressed in terms of square roots. Geometrically, this first fact implies that the regular 7-gon is not constructible with ruler and compass; the second fact implies that it is impossible in general to trisect an angle using ruler and compass.