Math 461 **Exam 2**
Spring 2023 **Tues Apr 4**

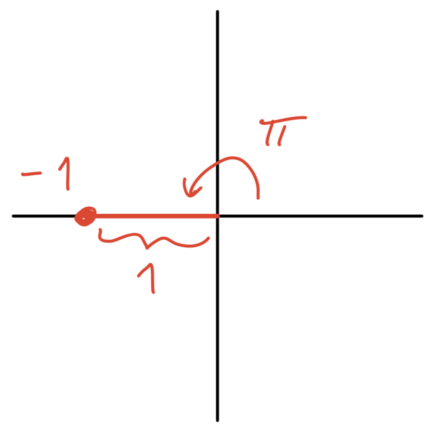
No electronic devices are allowed. There are 5 pages and each page is worth 6 points, for a total of 30 points.

Problem 1. Complex Numbers.

(a) Express -1 in polar form.

$$1 \cdot e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1 + i0 = -1$$

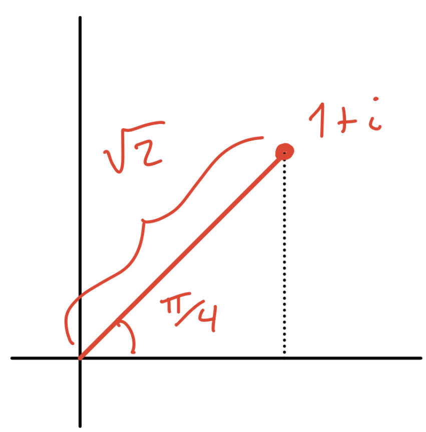
Picture:



(b) Express $1+i$ in polar form.

$$\sqrt{2} \cdot e^{i\pi/4} = \sqrt{2}\,(\cos(\pi/4) + i\sin(\pi/4)) = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = 1 + i.$$

Picture:



(c) Let $\omega = e^{i\theta}$ for some real $\theta \in \mathbb{R}$. Use Euler's formula to show that $\omega^* = \omega^{-1}$.

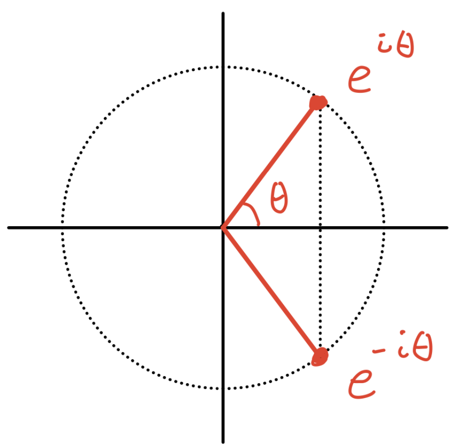
We have

$$\begin{aligned} \omega^* &= (e^{i\theta})^* \\ &= (\cos\theta + i\sin\theta)^* && \text{Euler's formula} \\ &= \cos\theta - i\sin\theta \end{aligned}$$

and

$$\begin{aligned} \omega^{-1} &= (e^{i\theta})^{-1} \\ &= e^{i(-\theta)} \\ &= \cos(-\theta) + i\sin(-\theta) && \text{Euler's formula} \\ &= \cos\theta - i\sin\theta. \end{aligned}$$

Picture:



Problem 2. Roots of Unity. Let $\omega = e^{i2\pi/6}$ so that

$$x^6 - 1 = (x-1)(x-\omega)(x-\omega^2)(x-\omega^3)(x-\omega^4)(x-\omega^5).$$

(a) Complete the sentence: For integers $k, \ell \in \mathbb{Z}$ we have $\omega^k = \omega^\ell$ if and only if …

$$k - \ell = 6n \text{ for some integer } n \in \mathbb{Z}.$$

(b) Find the complete factorization of $x^6 - 1$ over the real numbers. [Hint: Use part (a) and Problem 1(c) to group the non-real roots into complex conjugate pairs. Then use the fact that $\alpha = e^{i\theta}$ implies $\alpha\alpha^* = 1$ and $\alpha + \alpha^* = 2\cos\theta$.]

It follows from part (a) and 1(d) that $\omega^5 = \omega^{-1} = \omega^*$, hence

$$\begin{aligned} &= (x-\omega)(x-\omega^*) \\ &= x^2 - (\omega + \omega^*)x + 1 \\ &= x^2 - 2\cos(2\pi/6)x + 1 \end{aligned}$$

$$= x^2 - x + 1.$$

Similarly, we have $\omega^4 = \omega^{-2} = (\omega^2)^*$ and hence

$$\begin{aligned}(x-\omega^2)(x-\omega^4) &= (x-\omega^2)(x-(\omega^2)^*) \\ &= x^2 - (\omega^2 + (\omega^2)^*)x + 1 \\ &= x^2 - 2\cos(4\pi/6)x + 1 \\ &= x^2 + x + 1.\end{aligned}$$

Finally, since $\omega^3 = e^{i\pi} = -1$ we have

$$\begin{aligned}x^6 - 1 &= (x-1)(x-\omega)(x-\omega^2)(x-\omega^3)(x-\omega^4)(x-\omega^5) \\ &= (x-1)(x+1)(x-\omega)(x-\omega^5)(x-\omega^2)(x-\omega^4) \\ &= (x-1)(x+1)(x^2-x+1)(x^2+x+1).\end{aligned}$$

Problem 3. Roots of Other Complex Numbers.

(a) Find all of the third roots of $8i$. [Hint: Express $8i$ in polar form.]

Note that $e^{i\pi/2} = \cos(\pi/2) + i\sin(\pi/2) = 0 + i = i$. Hence

$$8i = 8 \cdot e^{i\pi/2}.$$

We are looking for $\alpha = re^{i\theta}$ such that

$$\begin{aligned}\alpha^3 &= 8i \\ (re^{i\theta})^3 &= 8e^{i\pi/2} \\ r^3 e^{i3\theta} &= 8e^{i\pi/2}.\end{aligned}$$

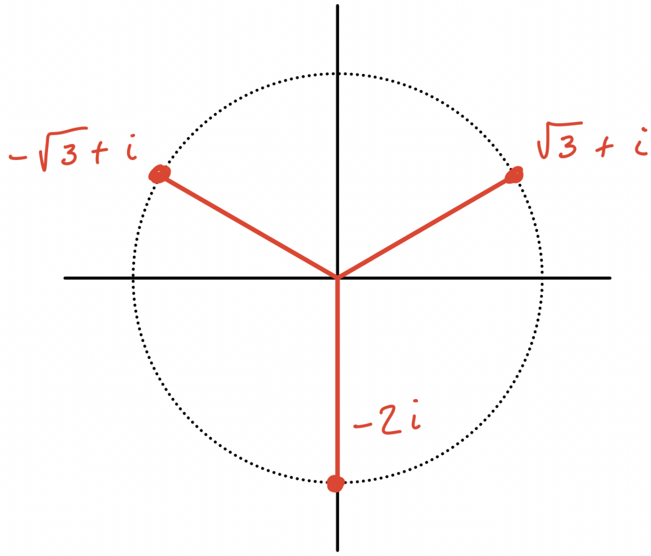
Comparing lengths gives $r^3 = 8$ and hence $r = 2$ because r is positive and real. Then comparing angles gives

$$\begin{aligned}e^{i3\theta} &= e^{i\pi/2} \\ 3\theta &= \pi/2 + 2\pi k \\ \theta &= \pi/6 + (2\pi/3)k\end{aligned}$$

for any integer $k \in \mathbb{Z}$. This corresponds to three angles $\theta = \pi/6, 5\pi/6, 9\pi/6$. Hence the third roots of $8i$ are

$$\begin{aligned}2 \cdot e^{i\pi/6} &= 2\left(\cos(\pi/6) + i\sin(\pi/6)\right) = \sqrt{3} + i, \\ 2 \cdot e^{i5\pi/6} &= 2\left(\cos(5\pi/6) + i\sin(5\pi/6)\right) = -\sqrt{3} + i, \\ 2 \cdot e^{i9\pi/6} &= 2\left(\cos(9\pi/6) + i\sin(9\pi/6)\right) = -2i.\end{aligned}$$

Picture:



(b) Use part (a) to completely factor $x^3 - 8i$ over the complex numbers.

From part (a) and Descartes' Theorem we have

$$x^3 - 8i = (x - (-2i))\left(x - (\sqrt{3} + i)\right)\left(x - (-\sqrt{3} + i)\right).$$

Alternatively, a few students observed that $(2i)^3 = -8i$ and then used the sum of cubes formula:

$$\begin{aligned} x^3 + y^3 &= (x + y)(x^2 - xy + y^2) \\ x^3 + (2i)^3 &= (x + 2i)(x^2 - (2i)x + (2i)^2) \\ x^3 - 8i &= (x + 2i)(x^2 - 2ix - 4). \end{aligned}$$

Then we can factor $x^2 - 2ix - 4$ using the quadratic formula:

$$x = \frac{2i \pm \sqrt{-4 + 16}}{2} = i \pm \sqrt{3}.$$

Problem 4. Abstract Conjugation. Let $\mathbb{E} \supseteq \mathbb{F}$ be a field extension and let $* : \mathbb{E} \to \mathbb{E}$ be any function with the following properties:

(1) $\alpha = \alpha^*$ if and only if $\alpha \in \mathbb{F}$,
(2) $\alpha^{**} = \alpha$,
(3) $(\alpha + \beta)^* = \alpha^* + \beta^*$,
(4) $(\alpha\beta)^* = \alpha^*\beta^*$.

(a) For any polynomial $f(x) \in \mathbb{F}[x]$ and constant $\alpha \in \mathbb{E}$ use the above properties to show that that $[f(\alpha)]^* = f(\alpha^*)$.

Consider a polynomial $f(x) = \sum a_k x^k$ with $a_k \in \mathbb{F}$ for all k. Then

$$[f(\alpha)]^* = \left(\sum a_k \alpha^k\right)^*$$

$$
\begin{aligned}
&= \sum \left(a_k \alpha^k\right)^* && (3)\\
&= \sum a_k^* (\alpha^*)^k && (4)\\
&= \sum a_k (\alpha^*)^k && (1)\\
&= f(\alpha^*).
\end{aligned}
$$

(b) For any polynomial $f(x) \in \mathbb{F}[x]$ and constant $\alpha \in \mathbb{E}$ use part (a) to show that $f(\alpha) = 0$ if and only if $f(\alpha^*) = 0$. [Hint: Property (2) implies that $\beta = \gamma$ if and only if $\beta^* = \gamma^*$.]

Remark: Suppose that $\beta, \gamma \in \mathbb{E}$ satisfy $\beta^* = \gamma^*$, so that $\beta^{**} = \gamma^{**}$. Then (2) implies $\beta = \gamma$. Also observe that property (1) implies $0^* = 0$. We will use these facts in our proof.

Proof: Consider a polynomial $f(x) \in \mathbb{F}[x]$ and a constant $\alpha \in \mathbb{E}$. Then we have

$$
\begin{aligned}
f(\alpha) = 0 \quad &\Longleftrightarrow \quad [f(\alpha)]^* = 0^* && \text{previous remark}\\
&\Longleftrightarrow \quad f(\alpha^*) = 0. && \text{part (a) and (1)}
\end{aligned}
$$

Problem 5. Complex Roots of Real Polynomials. Let $f(x) \in \mathbb{R}[x]$ be a real polynomial satisfying $f(1+i) = 0$. Thus from Descartes' Theorem we have

$$f(x) = (x - (1+i))g(x) \text{ for some complex polynomial } g(x) \in \mathbb{C}[x].$$

(a) Show that $g(1-i) = 0$. [Hint: Use Problem 4(b).]

Since $f(x)$ has real coefficients and $f(1+i) = 0$, Problem 4(b) implies that

$$0 = f((1+i)^*) = f(1-i).$$

But then

$$
\begin{aligned}
f(1-i) &= ((1-i) - (1+i))g(1-i)\\
0 &= (-2i)g(1-i)\\
0 &= g(1-i).
\end{aligned}
$$

(b) Use part (a) to show that $f(x) = (x^2 - 2x + 2)h(x)$ for some **real** polynomial $h(x) \in \mathbb{R}[x]$. You may assume the following result without proof: If $f(x) = p(x)h(x)$ with $f(x), p(x) \in \mathbb{R}[x]$ and $h(x) \in \mathbb{C}[x]$, then we must have $h(x) \in \mathbb{R}[x]$.

Since $g(1-i) = 0$, Descartes' Theorem implies that $g(x) = (x - (1-i))h(x)$ for some polynomial $h(x) \in \mathbb{C}[x]$. Then we have

$$
\begin{aligned}
f(x) &= (x - (1+i))(x - (1-i))h(x)\\
&= (x^2 - 2x + 2)h(x).
\end{aligned}
$$

Finally, since $f(x)$ and $x^2 - 2x + 2$ have real coefficients, we conclude that $h(x)$ has real coefficients.