MTH 461 **Spring 2020**
Writing Project **Drew Armstrong**

The system of *quaternions* was discovered by William Rowan Hamilton on the afternoon of Monday, October 16, 1843, as he walked along Royal Canal in Dublin. A quaternion is an abstract symbol of the form $a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, where $a, b, c, d \in \mathbb{R}$ are real numbers:

$$\mathbb{H} = \{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} : a, b, c, d \in \mathbb{R}\}.$$

The "imaginary units" $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are abstract symbols satisfying the following multiplication rules:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = \mathbf{-1}.$$

One can check that $(\mathbb{H}, +, \cdot, 0, 1)$ is a ring. However, it is not a commutative ring because (for example) we have $\mathbf{ij} = -\mathbf{k}$ and $\mathbf{ji} = -\mathbf{k} \neq \mathbf{k}$. Your assignment is to write a mathematical paper about the quaternions, including some of exposition and history, but focusing mainly on mathematical results. Here are some ideas:

- Given $\alpha = a+b\mathbf{k}+c\mathbf{j}+d\mathbf{k} \in \mathbb{H}$, we define the quaternion conjugate by $\alpha^* = a-b\mathbf{i}-c\mathbf{j}-d\mathbf{k}$. Show that $\alpha\alpha^* = \alpha^*\alpha = |\alpha| = (a^2 + b^2 + c^2 + d^2) \in \mathbb{R}$ and use this to show that every nonzero quaternion has a two-sided inverse: $\alpha\alpha^{-1} = \alpha^{-1}\alpha = 1$.
- For any $\alpha, \beta \in \mathbb{H}$ show that $(\alpha\beta)^* = \beta^*\alpha^*$. Then it follows from the previous remark that $|\alpha\beta| = |\alpha||\beta|$. Use this to show that if $m, n \in \mathbb{Z}$ can each be expressed as a sum of four integer squares, then mn can also be expressed as a sum of four integer squares.
- Explain how quaternions can be represented as 2×2 matrices with complex entries.
- Quaternions of the form $\mathbf{u} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ are called imaginary. We can also view $\mathbf{u}$ as the vector (u, v, w) in $\mathbb{R}^3$. Explain how the product of imaginary quaternions is related to the dot product and the cross product of vectors.
- If $\mathbf{u}$ is imaginary of length 1, show that $\mathbf{u}^2 = -1$. It follows that the polynomial $x^2 + 1 \in \mathbb{H}[x]$ of degree 2 has infinitely many roots in $\mathbb{H}$. Why does this not contradict Descartes' Factor Theorem? [Hint: $\mathbb{H}$ is not commutative.]
- Show that every quaternion $\alpha \in \mathbb{H}$ can be written in polar form as $\alpha = |\alpha|(\cos\theta + \mathbf{u}\sin\theta)$, where $\mathbf{u}$ is imaginary of length 1.
- For any $\alpha, \mathbf{x} \in \mathbb{H}$ with $\alpha \neq 0$ and $\mathbf{x}$ imaginary, show that $\alpha^{-1}\mathbf{x}\alpha$ is imaginary.
- Suppose that $\alpha = \cos\theta + \mathbf{u}\sin\theta$ where $\mathbf{u}$ is imaginary of length 1, and let $\mathbf{x}$ be any imaginary quaternion. Recall that we can also think of $\mathbf{u}$ and $\mathbf{x}$ as vectors in $\mathbb{R}^3$. Explain why $\alpha^{-1}\mathbf{x}\alpha$ corresponds to the **rotation** of $\mathbf{x}$ around the axis $\mathbf{u}$ by angle 2θ.
- Let $\mathbf{u} \in \mathbb{H}$ be imaginary of length 1 and let $\theta \in \mathbb{R}$ be real. Explain why it makes sense to define the exponential notation $e^{\theta\mathbf{u}} = \cos\theta + \mathbf{u}\sin\theta$.
- Consider the following set of 24 quaternions: $\{\pm 1, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}, (\pm 1 \pm \mathbf{i} \pm \mathbf{j} \pm \mathbf{k})/2\}$. Explain how this set is related to a regular tetrahedron. [Hint: A regular tetrahedron has 12 rotational symmetries.]