Writing Project due Mon.

HW4 : TBA.

Exams ???

Modern algebra is based on abstract structures such as rings, fields, polynomials, vector spaces, groups, etc.

Back to $\mathbb{C}$.

We know Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

but the proof was mysterious. Now I will make this less mysterious by showing you the modern (20th century) interpretation.

Idea:

complex number $\equiv$ some kind of function.

We will view $\mathbb{C}$ as a vector space over the real numbers.

DEF: Let $\mathbb{F}$ be a field. A <u>vector space</u> over $\mathbb{F}$ is a set V (of vectors) with two operations

$u, v \in V \rightsquigarrow u + v \in V$ vector addition

$\alpha \in \mathbb{F}, v \in V \rightsquigarrow \alpha v \in V$ scalar multiplication

satisfying the following rules:

- $u + v = v + u \quad \forall u, v \in V$
- $u + (v + w) = (u + v) + w \quad \forall u, v, w \in V$
- $\exists 0 \in V, \ u + 0 = u \quad \forall u \in V$
- $\forall u \in V, \exists -u \in V, \ u + (-u) = 0$
- $(\alpha\beta)u = \alpha(\beta u) \quad \forall \alpha, \beta \in \mathbb{F}, u \in V$
- $\alpha(u + v) = \alpha u + \alpha v \quad \forall \ldots$
- $(\alpha + \beta)u = \alpha u + \beta u \quad \forall \ldots$
- $1u = u \quad \forall \ldots$

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Example: Cartesian space

$\mathbb{R}^n$ = ordered n-tuples of real numbers.

$$= \{(x_1, x_2, \dots, x_n) : x_i \in \mathbb{R}\}$$

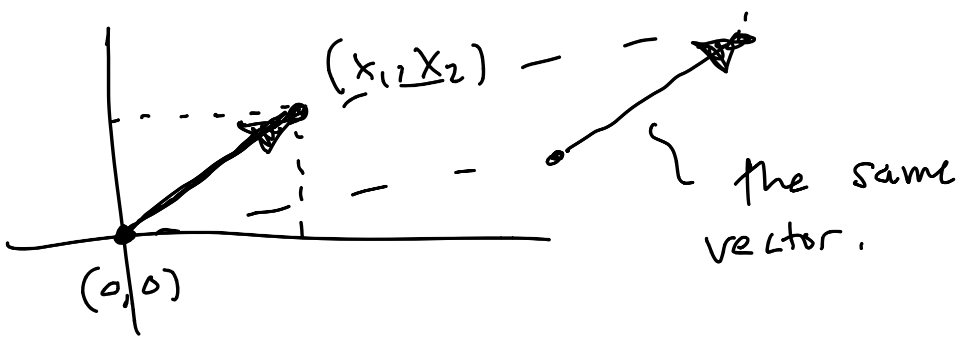
This is a V.S. over $\mathbb{R}$ with operations

$$(x_1, \dots, x_n) + (y_1, \dots, y_n) := (x_1 + y_1, \dots, x_n + y_n)$$

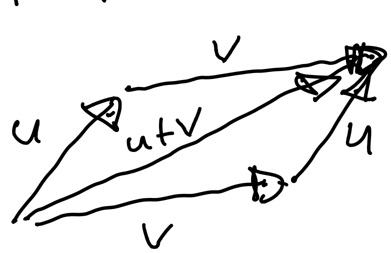
$$\alpha(x_1, \dots, x_n) := (\alpha x_1, \alpha x_2, \dots, \alpha x_n)$$

Exercise: Check this!

Picture: The vector (x_1, x_2)

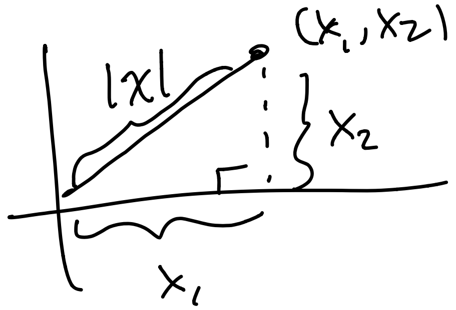


Reason: Vectors add head-to-tail:



$u + v = v + u$

Furthermore, we can define the <u>length</u> of a vector in terms of Pythag. Thm.:



$$|x| = \sqrt{x_1^2 + x_2^2}$$

More generally, for $x = (x_1, x_2, \dots, x_n)$, we define the length as

$$|x| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}.$$

Remark: If you believe in n-dim space, then this is a theorem not a definition.

Back to $\mathbb{C}$.

We can view $\mathbb{C}$ as the vector space $\mathbb{R}^2$ by identifying

$$\begin{array}{ccc} \mathbb{C} & \longleftrightarrow & \mathbb{R}^2 \\ a + bi & & (a, b) \end{array}$$

Why would we bother?

DEF: Let V be v.s. over $\mathbb{F}$.
We say function $f: V \to V$ is "$\mathbb{F}$-linear" if for all $\alpha, \beta \in \mathbb{F}$ & $u, v \in V$ we have

$$\boxed{f(\alpha u + \beta v) = \alpha f(u) + \beta f(v).}$$

"preserves linear combinations"

NICE: A linear function

$$f: \mathbb{R}^n \to \mathbb{R}^n$$

can be represented as a $n \times n$ matrix with real entries.

How? Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be linear. and for all i let f_i be the <u>vector</u>

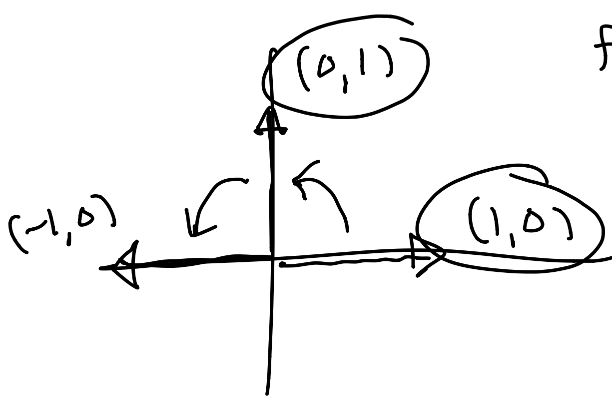
$$f_i := f(0, 0, \cdots 0, 1, 0, \cdots, 0)$$

ith positions

Then we express f as the matrix

$$[f] = \begin{pmatrix} | & | & & | \\ f_1 & f_2 & \cdots & f_n \\ | & | & & | \end{pmatrix}.$$

Example: Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be the "rotate c.c.w. by $90°$" function. What is the matrix?



$f_1 = f(1,0) = ? = (0,1)$

$f_2 = f(0,1) = ? = (-1,0)$

The matrix is

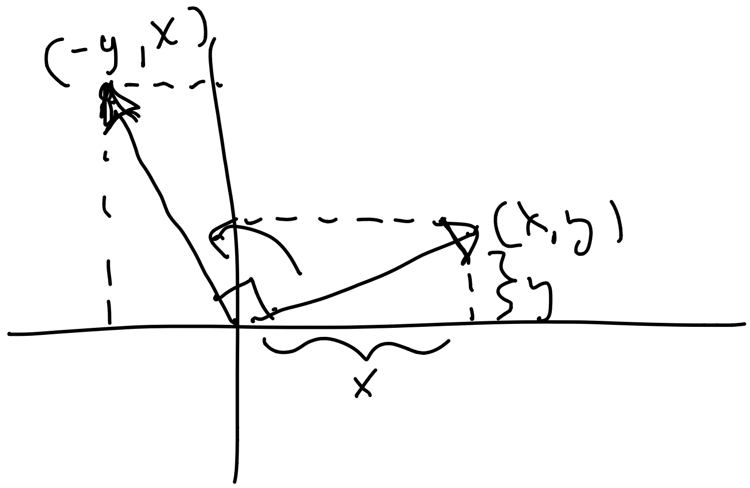
$$[f] = \begin{pmatrix} | & | \\ f_1 & f_2 \\ | & | \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Thus, we can rotate any vector (x,y) c.c.w. by $90°$ as follows:

$$f(x,y) = [f]\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 - y \\ x + 0 \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$$

Picture:

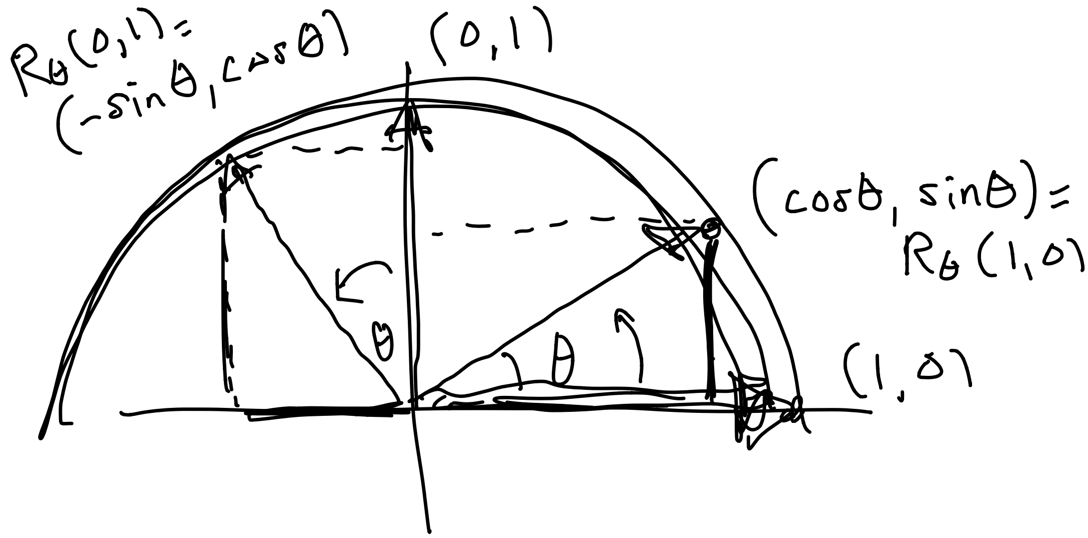


More generally, let R_θ be the 2×2 real matrix that rotates any vector c.c.w. by angle θ. Find it!

We only need to compute

$R_\theta(1,0)$ & $R_\theta(0,1)$

Picture:



We conclude that the rotation matrix is

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Important Observation:

Multiplication of matrices

$\|$

Composition of (linear) functions.

In fact, this is how we <u>define</u> the multiplication of matrices!

As a consequence, we obtain the angle sum identities "for free"!

<u>Proof</u>: for any angles $\alpha, \beta \in \mathbb{R}$,

$$
\begin{aligned}
R_{\alpha+\beta} &= \text{rotate by } \alpha+\beta \\
&= \text{rotate by } \beta \ \underline{\text{then}} \ \text{rotate by } \alpha \\
&= R_\alpha R_\beta
\end{aligned}
$$

Explicitly, we get

$$
\begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}
$$

Expand & see what you get!

MAGIC!