Hello! ~

# Algebra Before 1830.

Carl Friedrich Gauss

Disquisitiones Arithmeticae

~ 1800

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Gauss-Wantzel Theorem

1800 1832

"Solved Trigonometry"

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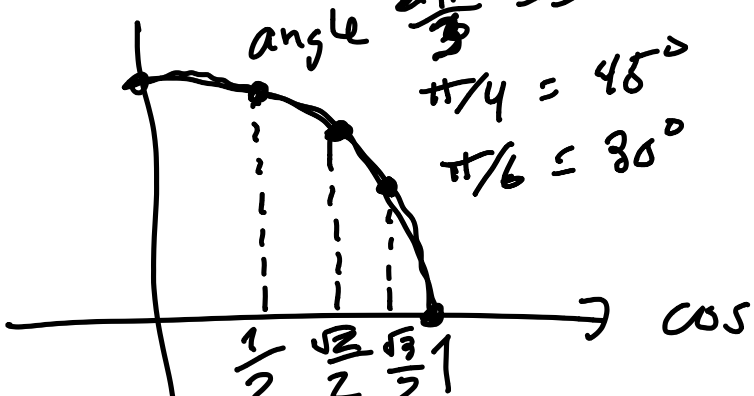
angle $\frac{\pi}{3} = 60^\circ$

$\pi/4 = 45^\circ$

$\pi/6 = 30^\circ$

What about other angles?




$\sin = \sqrt{\cos^2 - 1}$

$\frac{\sqrt{0}}{2}$ $\frac{\sqrt{1}}{2}$ $\frac{\sqrt{2}}{2}$ $\frac{\sqrt{3}}{2}$ $\frac{\sqrt{4}}{2}$

Problem: Find formulas for the numbers $\cos\left(\frac{2\pi}{n}\right)$ & $\cos\left(\frac{2\pi k}{n}\right)$ for all integers $0 \le k < n$.

Recall: $\cos(2\theta) = 2\cos^2\theta - 1$

$$\cos\theta = 2\cos^2\left(\frac{\theta}{2}\right) - 1$$

$$\cos^2\left(\frac{\theta}{2}\right) = \frac{1+\cos\theta}{2} = \frac{2+2\cos\theta}{4}$$

$$\cos\left(\frac{\theta}{2}\right) = \frac{\pm 1}{2}\sqrt{2+2\cos\theta}.$$

Example:

$$\cos(\pi) = -1$$

$$\cos\left(\frac{\pi}{2}\right) = \frac{1}{2}\sqrt{2-2} = 0$$

$$\cos\left(\frac{\pi}{4}\right) = \frac{1}{2}\sqrt{2+0} = \sqrt{2}/2$$

$$\cos\left(\frac{\pi}{8}\right) = \frac{1}{2}\sqrt{2+\sqrt{2}}$$

$$\cos\left(\frac{\pi}{16}\right) = \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}.$$

Observe: $\cos\left(\frac{\pi}{2^n}\right)$ has nested square roots in the formula.

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What next?

Express $\cos(\theta/3)$ in terms of $\cos\theta$?

Recall: $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$

$$\Longrightarrow \cos\left(\frac{\theta}{3}\right) = \sqrt[3]{\ldots} + \sqrt[3]{\ldots}$$

...

formulas involving $\cos\theta$ and some square roots.

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It gets worse.

There is no general algebraic formula for $\cos(\theta/5)$ in terms of $\cos\theta$.

Nevertheless, we know quite a lot about the numbers $\cos\left(\frac{2\pi k}{n}\right)$ where $k, n \in \mathbb{Z}$.

Theorem:

- $\cos\left(\frac{2\pi}{n}\right) \in \mathbb{Q}$ if and only if $n = 1, 2, 3, 4, 6$
- $\cos\left(\frac{2\pi}{n}\right)$ can be expressed using a <u>single</u> square root if and only if $n = 5, 8, 10, 12$

Some you know already:

$\cos\left(\frac{2\pi}{8}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ ✓

$\cos\left(\frac{2\pi}{12}\right) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ ✓

What about $\cos\left(\frac{2\pi}{5}\right)$ & $\cos\left(\frac{2\pi}{10}\right)$?

Remark: The formulas for $\cos\left(\frac{2\pi}{5}\right)$ & $\cos\left(\frac{2\pi}{10}\right)$ are "accidentally good." The ancient Greeks knew this in a different form. They knew that a regular pentagon is constructible using straightedge & compass (i.e. using Euclidean Geometry)

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Using complex numbers, it's not so hard. Let $\omega = e^{2\pi i/5}$, so

$$\omega = \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right)$$
$$\omega^{-1} = \cos\left(\frac{2\pi}{5}\right) - i\sin\left(\frac{2\pi}{5}\right)$$

complex conjugates

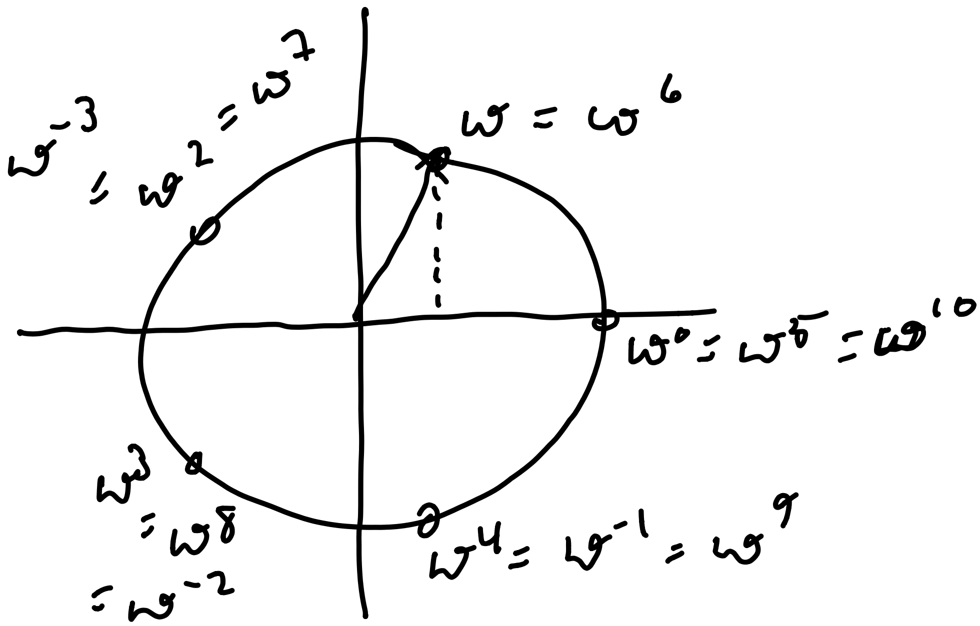
Recall: $(\omega^k)^* = \omega^{-k}$

$$\omega + \omega^{-1} = 2\cos\left(\frac{2\pi}{5}\right) = z$$

Also recall:

$$1+\omega+\omega^2+\omega^3+\omega^4=0$$

$$\omega^0+\omega^1+\omega^2+\underset{\omega^{-2}}{\cancel{\omega^3}}+\underset{\omega^{-1}}{\cancel{\omega^4}}=0$$




$$\boxed{\omega^2+\omega^1+\omega^0+\omega^{-1}+\omega^{-2}=0}$$

$$\boxed{\omega^1+\omega^{-1}=2\cos\left(\frac{2\pi}{5}\right)=z}$$

$$z^2=(\omega+\omega^{-1})(\omega+\omega^{-1})$$

$$=\omega^2+2\cancel{\omega\omega^{-1}}+\omega^{-2}$$

$$=\omega^2+2+\omega^{-2}$$

$$
\begin{array}{rcccccccccc}
z^0 = & & & & & 1 & & & & \\
z^1 = & & & w & + & 0 & + & w^{-1} & & \\
z^2 = & w^2 & + & 0 & + & 2 & + & 0 & + & w^{-2} \\
\hline
\end{array}
$$

$$1+z+z^2 = w^2 + w^1 + 3 + w^{-1} + w^{-2}$$

$$= \cancel{\left(w^2+w^1+1+w^{-1}+w^{-2}\right)} + 2$$

$$= 0 + 2 = 2$$

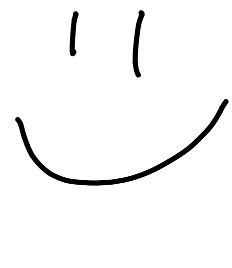
$$1 + z + z^2 = 2$$

$$z^2 + z - 1 = 0$$

$$z = \frac{-1 \pm \sqrt{5}}{2}$$ choose positive

$$2\cos\left(\frac{2\pi}{5}\right) = \frac{-1+\sqrt{5}}{2}$$

$$\cos\left(\frac{2\pi}{5}\right) = \frac{-1+\sqrt{5}}{4}$$



$\cos\left(\frac{2\pi}{10}\right)$ on HW . . .

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Every other value of $\cos\left(\frac{2\pi k}{n}\right)$ is worse than this.

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Recall the function

$$\phi(n) = \#\text{ integers } 0 \leq k < n \text{ such that } \gcd(k,n) = 1$$

$$\phi(12) = \#\{1, \cancel{2}, \cancel{3}, \cancel{4}, 5, \cancel{6}, 7, \cancel{8}, \cancel{9}, \cancel{10}, 11\}$$

$$= 4.$$

Later we will have a sort of reasonable formula for $\phi(n)$.

Theorem: Suppose

$\phi(n) = m_1 m_2 m_3 \cdots m_k$

Then (Gauss) we can express $\cos\left(\frac{2\pi}{n}\right)$ in terms of integers by taking $m_1^{th}$ root, then $m_2^{th}$ root, then $m_3^{th}$ root $\cdots$ then $m_k^{th}$ root.

Special Case: If $\phi(n) = 2^k$ then $\cos\left(\frac{2\pi}{n}\right)$ can be expressed in terms of $\mathbb{Z}$ using $k$ nested square roots.

NEXT TIME!