Math 461 **Spring 2020**
Homework 2 **Drew Armstrong**

Problem 1. One Real Root. Consider a polynomial $x^3 + px + q$ with real coefficients $p, q \in \mathbb{R}$ satisfying $p > 0$. We will show that this polynomial has exactly one real root.

(a) From the Intermediate Value Theorem we know that there exists a real root $f(r) = 0$. In this case use long division to show that
$$f(x) = (x - r)(x^2 + rx + p + r^2).$$

(b) Show that $x^2 + rx + p + r^2$ has no real roots. [Hint: Consider the discriminant.]

Problem 2. Coefficients Versus Roots. Let $\mathbb{F}$ be a field and suppose that the polynomial $f(x) = x^3 + ax^2 + bx + c \in \mathbb{F}[x]$ has three roots $r, s, t \in \mathbb{F}$.

(a) Find formulas for a, b, c in terms of r, s, t.

(b) Find a formula for $r^2 + s^2 + t^2$ in terms of a, b, c. [Hint: Square $r + s + t$.]

Problem 3. Uniqueness of Roots. Let $f(x) \in \mathbb{F}[x]$ be a polynomial with coefficients in a field $\mathbb{F}$. Suppose that there exist numbers $a_1, \ldots, a_r \in \mathbb{F}$ and $b_1, \ldots, b_s \in \mathbb{F}$ such that
$$f(x) = (x - a_1)(x - a_2) \cdots (x - a_r) = (x - b_1)(x - b_2) \cdots (x - b_s).$$

(a) Prove that $r = s$. [Hint: Degree.]

(b) Prove that the roots can be re-indexed so that $a_i = b_i$ for all i. [Hint: Consider $f(a_i)$.]

Problem 4. Cardano's Formula. Cardano's formula applied to $x^3 + 6x - 20 = 0$ gives
$$x = \sqrt[3]{10 + \sqrt{108}} + \sqrt[3]{10 - \sqrt{108}}.$$
Observe that $\sqrt{108} = 6\sqrt{3}$. Try to find some integers $a, b, c, d \in \mathbb{Z}$ such that
$$(a + b\sqrt{3})^3 = 10 + \sqrt{108} \quad \text{and} \quad (c + d\sqrt{3})^3 = 10 - \sqrt{108}.$$
Then use your answer to prove that
$$\sqrt[3]{10 + \sqrt{108}} + \sqrt[3]{10 - \sqrt{108}} = 2.$$

Problem 5. A Prime Cubic Polynomial. We will give a rigorous proof that the polynomial $f(x) = x^3 + x + 1$ is a prime element of the ring $\mathbb{Q}[x]$.

(a) Suppose that we have $f(a/b) = 0$ for some integers $a, b \in \mathbb{Z}$. By reducing a/b to lowest terms we may assume that a and b have no common prime factors. In this case show that $a = \pm 1$ and $b = \pm 1$. [Hint: If $p|a$ for some prime $p \in \mathbb{Z}$, then $p|b^3$ and hence $p|b$.]

(b) Use part (a) to show that $f(x)$ has no roots in $\mathbb{Q}$.

(c) Show that every polynomial in $\mathbb{Q}[x]$ of degree 1 has a root in $\mathbb{Q}$.

(d) If $f(x) \in \mathbb{Q}[x]$ is **not** prime then we can write $f(x) = g(x)h(x)$ for some polynomials $g(x), h(x) \in \mathbb{Q}[x]$ with $\deg(g) > 0$ and $\deg(h) > 0$. Show that one of $g(x)$ or $h(x)$ must have degree 1 and use this to obtain a contradiction.

Problem 6. Complex Conjugation. Consider the field of real numbers:
$$\mathbb{C} = \{a + b\sqrt{-1} : a, b \in \mathbb{R}\}.$$
We define *complex conjugation* $* : \mathbb{C} \to \mathbb{C}$ by the following formula:
$$(a + b\sqrt{-1})^* := a - b\sqrt{-1}.$$

(a) For all $\alpha \in \mathbb{C}$ show $\alpha^* = \alpha$ if and only if $\alpha \in \mathbb{R}$.
(b) For all $\alpha, \beta \in \mathbb{C}$ show that $(\alpha + \beta)^* = \alpha^* + \beta^*$ and $(\alpha\beta)^* = \alpha^*\beta^*$.
(c) For all real polynomials $f(x) \in \mathbb{R}[x]$ and complex numbers $\alpha \in \mathbb{C}$ show that

$$f(\alpha)^* = f(\alpha^*).$$

(d) Use part (c) to show that complex roots of real polynomials come in conjugate pairs. It follows that any real polynomial has an **even number** of complex roots.