Current Topic: "The Fundamental Theorem of Algebra," which says that <u>every</u> polynomial $f(x) \in \mathbb{C}[x]$ <u>has some root</u> in $\mathbb{C}$, say $\alpha_1 \in \mathbb{C}$.

Descartes: $\underset{\deg n}{f(x)} = (x-\alpha_1)\underset{\deg n-1}{g(x)}$.

But then for the same reason, $g(x)$ has a complex root, say $\alpha_2 \in \mathbb{C}$.

Descartes: $\underset{\deg n}{f(x)} = (x-\alpha_1)(x-\alpha_2)\underset{\deg n-2}{h(x)}$.

Continuing ...

$f(x) = (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)\,c$

for some $\alpha_1, \ldots, \alpha_n, c \in \mathbb{C}$.

Equivalent statements of F.T.A.

- every $f(x) \in \mathbb{C}[x]$ has a root in $\mathbb{C}$.
- every $f(x) \in \mathbb{C}[x]$ splits over $\mathbb{C}$.
- every <u>prime</u> polynomial in $\mathbb{C}[x]$ has degree 1.

Our current goal is to prove this. The proof is not easy. Indeed, Leonhard Euler tried & failed to prove it. Lagrange fixed some problems with Euler's approach, then Laplace (~1815) finally gave a correct version. We will follow Laplace's proof, but it has some pre-requisites!

History of F.T.A.

In the 1600s people tried to compute antiderivatives / integrals.

Fermat: $$\int x^n \, dx = \begin{cases} \frac{1}{n+1} x^{n+1} & n \neq -1 \\ \ln|x| & n = -1 \end{cases}$$

From this we can integrate any polynomial $\in \mathbb{R}[x]$:

$$\int \sum_{k \geq 0} a_k x^k \, dx = \sum_{k \geq 0} a_k \frac{1}{k+1} x^{k+1}$$:)

Next Problem: Integrate "rational functions" of the form $f(x)/g(x)$ where $f(x), g(x) \in \mathbb{R}[x]$:

$$\int \frac{f(x)}{g(x)}\,dx = ?$$

A few "basic" examples:

- $\displaystyle\int \frac{1}{x+a}\,dx = \ln|x+a| \qquad a \in \mathbb{R}$
- $\displaystyle\int \frac{1}{x^2+b^2}\,dx = \frac{1}{b}\arctan\left(\frac{x}{b}\right) \qquad b \in \mathbb{R}$
- $\displaystyle\int \frac{x}{x^2+c^2}\,dx = \frac{1}{2}\ln\left(x^2+c^2\right) \qquad c \in \mathbb{R}$

It was realized early on that these 3 rules are enough to integrate <u>any</u> rational function as long as the denominator can be factored into deg 1 & deg 2 polynomials.

Method of "Partial Fractions."

Example: Compute $\int \frac{1}{(x-1)(x^2+4)}\,dx$.

Idea: $\exists\, A, B, C \in \mathbb{R}$ such that

$$\frac{1}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}.$$

Find Them!

$$\frac{1}{(x-1)(x^2+4)} = \frac{A(x^2+4) + (Bx+C)(x-1)}{(x-1)(x^2+4)}$$

$$1 \equiv A(x^2+4) + (Bx+C)(x-1)$$

$$1 = Ax^2 + Bx^2 - Bx + Cx + 4A - C$$

$$\underset{1+0x+0x^2}{1} = (A+B)x^2 + (C-B)x + (4A-C)$$

$$\Rightarrow \begin{cases} A+B=0 \\ C-B=0 \\ 4A-C=1 \end{cases}$$

$A + C = 0$

$C = -A$

$4A + A = 1$

$5A = 1$

$A = 1/5$

$B = -A = -1/5$

$C = -A = -1/5$.

Conclude:

$$\frac{1}{(x-1)(x^2+4)} = \frac{1}{5}\frac{1}{(x-1)} - \frac{1}{5}\frac{x+1}{x^2+4}$$

$$= \frac{1}{5}\left[\underbrace{\frac{1}{x-1} - \frac{1}{x^2+4} - \frac{x}{x^2+4}}_{\text{each is one of the three basic forms.}}\right]$$

$$\int \frac{1}{(x-1)(x^2+4)}\,dx = \frac{1}{5}\left[\ln|x-1| - \frac{1}{2}\arctan\left(\frac{x}{2}\right) - \frac{1}{2}\ln(x^2+4)\right]$$

Note that the F.T.A. implies that every $f(x) \in \mathbb{R}[x]$ factors into deg 1 & deg 2 polynomials. Indeed,

$$f(x) \in \mathbb{R}[x] \subseteq \mathbb{C}[x]$$

$$\underset{\text{FTA}}{\Longrightarrow} \quad f(x) = (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$$

$$\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{C}.$$

But since f has real coeffs, the non-real complex roots come in conjugate pairs.

$$f(x) = \overbrace{(x-a_1)\cdots(x-a_k)}^{\text{deg } 1}$$
$$\cdot\, (x-\alpha_1)(x-\alpha_1^*)\cdots(x-\alpha_l)(x-\alpha_l^*)$$

$$\underbrace{\left(x^2-\underset{\text{real}}{(\alpha_1+\alpha_1^*)}x-\underset{\text{real}}{\alpha_1\alpha_1^*}\right)}_{\text{real of deg } 2}$$

That's good enough to integrate any rational function (in principle).

"Partial Fractions"

We can compute partial fractions over $\mathbb{Z}$ and $\mathbb{F}[x]$.

"Euclidean Domains"

(Integral domains that have division with remainder)

DEF: A Euclidean domain is a (commutative) ring R such that

- R is an "integral domain":

  $ab = 0 \implies a = 0$ or $b = 0$.

- There is a "norm function" $N : R \setminus 0 \to \mathbb{N}$ such that $\forall\, a, b \in R,\ b \neq 0,\ \exists\, q, r \in R$ such that

$$\begin{cases} a = qb + r, \\ r = 0 \text{ or } N(r) < N(b). \end{cases}$$

Examples: $R = \mathbb{Z}$ with $N(a) = |a|$.

$R = \mathbb{F}[x]$ with $N(f) = \deg(f)$.

The whole apparatus of "unique prime factorization" holds in a general Euclidean domain.

Similarly, we can compute "partial fraction expansions" over any Euclidean domain.

Example: Expand $\frac{8}{15}$ into partial fractions.

Factor the denominator:

$$\frac{8}{15} = \frac{8}{3 \cdot 5} = \frac{?}{3} + \frac{?}{5}.$$

Now what?

PAUSE.