Math 461 **Spring 2015**
The Cubic Formula **Drew Armstrong**

The full cubic formula is too complicated to write on the board, so I've typed it here. Recall that the solution to the depressed cubic

$$x^3 + px + q = 0$$

is given by Cardano's formula

$$x = \sqrt[3]{-\left(\frac{q}{2}\right) + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{-\left(\frac{q}{2}\right) - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$$

Now consider the general cubic equation

$$ax^3 + bx^2 + cx + d = 0.$$

We assume that $a \neq 0$, so this is equivalent to the equation

$$x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0.$$

To convert this into a depressed cubic, we make the substitution $x = y - \frac{b}{3a}$ [how did we find this substitution?] to obtain

$$y^3 + 0y^2 + \left(\frac{3ac - b^2}{3a^2}\right) y + \left(\frac{27a^2d - 9abc + 2b^3}{27a^3}\right) = 0.$$

Finally, using Cardano's formula to solve for y gives

$$x = -\frac{b}{3a} + \sqrt[3]{\frac{27a^2d - 9abc + 2b^3}{54a^3} + \frac{\sqrt{3}}{18a^2}\sqrt{27a^2d^2 - 18abcd + 4ac^3 + 4b^3d - b^2c^2}}$$
$$+ \sqrt[3]{\frac{27a^2d - 9abc + 2b^3}{54a^3} - \frac{\sqrt{3}}{18a^2}\sqrt{27a^2d^2 - 18abcd + 4ac^3 + 4b^3d - b^2c^2}}.$$

I can't really make it look simpler than that. Are you surprised that no one taught this to you in high school?

One thing we notice from the formula is that the expression under the square root sign,

$$\Delta := 27a^2d^2 - 18abcd + 4ac^3 + 4b^3d - b^2c^2,$$

seems to be important. This is called the discriminant of the cubic equation, and it governs the qualitative behavior of the roots. Assuming that the coefficients a, b, c, d are **real** numbers, then $\Delta > 0$ means that the equation has 3 distinct real roots, $\Delta < 0$ means that there is 1 real root and 2 complex roots, and $\Delta = 0$ means that there is a multiple root, and all of the roots are real.

We will discuss the complex roots soon.