Math 461 **Spring 2015**
Homework 6 **Drew Armstrong**

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1. Symmetric Functions. Consider the elementary symmetric functions

$$\begin{aligned} e_1 &= r + s + t \\ e_2 &= rs + rt + st \\ e_3 &= rst. \end{aligned}$$

They are called elementary because every other symmetric function can be expressed in terms of them. Express the following symmetric functions in terms of e_1, e_2, e_3.

(a) $r^2 + s^2 + t^2$
(b) $r^3 + s^3 + t^3$
(c) $\frac{1}{r^2} + \frac{1}{s^2} + \frac{1}{t^2}$

For part (a), we note that the leading term of $r^2 + s^2 + t^2$ is r^2. This matches the leading term of e_1^2. Subtracting gives

$$\begin{aligned} (r^2 + s^2 + t^2) - e_1^2 &= (r^2 + s^2 + t^2) - (r^2 + s^2 + t^2 + 2rs + 2rt + 2st) \\ &= -2(rs + rt + st) \\ &= -2e_2. \end{aligned}$$

Hence $r^2 + s^2 + t^2 = e_1^2 - 2e_2$.

For part (b), we note that the leading term of $r^3 + s^3 + t^3$ is r^3, which matches the leading term of $e_1^3 = r^3 + s^3 + t^3 + 3r^2s + 3rs^2 + 3r^2t + 3rt^2 + 3s^2t + 3st^2 + 6rst$. Subtracting gives

$$(r^3 + s^3 + t^3) - e_1^3 = -3r^2s - 3rs^2 - 3r^2t - 3rt^2 - 3s^2t - 3st^2 - 6rst.$$

Note that the new leading term is $-3r^2s$, which matches the leading term of

$$-3e_1e_2 = -3(r + s + t)(rs + rt + st) = -3r^2s - 3rs^2 - 3r^2t - 3rt^2 - 3s^2t - 3st^2 - 9rst.$$

Subtracting gives

$$(r^3 + s^3 + t^3) - e_1^3 + 3e_1e_2 = 3rst = 3e_3.$$

Hence $r^3 + s^3 + t^3 = e_1^3 - 3e_1e_2 + 3e_3$.

For part (c), note that

$$\frac{1}{r^2} + \frac{1}{s^2} + \frac{1}{t^2} = \frac{r^2s^2 + r^2t^2 + s^2t^2}{r^2s^2t^2} = \frac{r^2s^2 + r^2t^2 + s^2t^2}{e_3^2}.$$

Now our goal is to express the numerator in terms of e_1, e_2, e_3. The leading term of the numerator is r^2s^2, which matches the leading term of e_2^2. Subtracting gives

$$\begin{aligned} (r^2s^2 + r^2t^2 + s^2t^2) - e_2^2 &= (r^2s^2 + r^2t^2 + s^2t^2) - (r^2s^2 + r^2t^2 + s^2t^2 + 2r^2st + 2rs^2t + 2rst^2) \\ &= -2r^2st - 2rs^2t - 2rst^2. \end{aligned}$$

The new leading term is $-2r^2st$, which matches the leading term of

$$-2e_1e_3 = -2(r + s + t)(rst) = -2r^st - 2rs^2t - 2rst^2.$$

Subtracting gives

$$(r^2s^2 + r^2t^2 + s^2t^2) - e_2^2 + 2e_1e_3 = 0.$$

Hence

$$\frac{1}{r^2}+\frac{1}{s^2}+\frac{1}{t^2}=\frac{e_2^2-2e_1e_3}{e_3^2}.$$

2. Application. Suppose that the polynomial x^3+px+q has roots r,s,t. Find the polynomial (with leading coefficient 1) whose roots are rs,rt,st. The coefficients must be expressed in terms of p and q. [Hint: The polynomial is $(x-rs)(x-rt)(x-st)$.]

We are given that

$$\begin{aligned} x^3+px+q &= (x-r)(x-s)(x-t) \\ &= x^3-(r+s+t)x^2+(rs+rt+st)x-(rst), \end{aligned}$$

and hence $e_1=r+s+t=0$, $e_2=rs+rt+st=p$, and $e_3=rst=-q$. The polynomial we are looking for is

$$\begin{aligned} (x-rs)(x-rt)(x-st) &= x^3-(rs+rt+st)x^2+(r^2st+rs^2t+rst^2)x-(r^2s^2t^2) \\ &= x^3-e_2x^2+e_1e_3x-e_3^2 \\ &= x^3-px^2+0x-(-q)^2 \\ &= x^3-px^2-q^2. \end{aligned}$$

3. Discrete Fourier Transform. Let $\omega=e^{2\pi i/3}$. In class I claimed that

$$\begin{pmatrix}1&1&1\\1&\omega&\omega^2\\1&\omega^2&\omega\end{pmatrix}^{-1}=\frac{1}{3}\begin{pmatrix}1&1&1\\1&\omega^2&\omega\\1&\omega&\omega^2\end{pmatrix}.$$

Verify that this is true.

First note that $\omega^3=1$, $\omega^4=\omega$, and $1+\omega+\omega^2=0$. Then we have

$$\begin{pmatrix}1&1&1\\1&\omega&\omega^2\\1&\omega^2&\omega\end{pmatrix}\begin{pmatrix}1&1&1\\1&\omega^2&\omega\\1&\omega&\omega^2\end{pmatrix}=\begin{pmatrix}1+1+1&1+\omega^2+\omega&1+\omega+\omega^2\\1+\omega+\omega^2&1+\omega^3+\omega^3&1+\omega^2+\omega^4\\1+\omega^2+\omega&1+\omega^4+\omega^2&1+\omega^3+\omega^3\end{pmatrix}=\begin{pmatrix}3&0&0\\0&3&0\\0&0&3\end{pmatrix}.$$

This is just what we wanted to show.

4. Example of Lagrange's Method. In this problem we will find the full solution to the cubic equation $x^3-6x-6=0$. [Compare to Exam 1 Problem 3.] Let r_1,r_2,r_3 be the three complex solutions and note that $e_1=r_1+r_2+r_3=0$, $e_2=r_1r_2+r_1r_3+r_2r_3=-6$, $e_3=r_1r_2r_3=6$. Let $\omega=e^{2\pi i/3}$, and define

$$\begin{aligned} s_1 &= r_1+r_2+r_3 \\ s_2 &= r_1+\omega r_2+\omega^2 r_3 \\ s_3 &= r_1+\omega^2 r_2+\omega r_3. \end{aligned}$$

(a) We saw in class that $s_2^3+s_3^3=2e_1^3-9e_1e_2+27e_3=162$ and $s_2s_3=e_1^2-3e_2=18$. Use this information to compute the values of s_2^3 and s_3^3.

(b) Let s_2 and s_3 be the positive real cube roots of the values for s_2^3 and s_3^3 that you computed above (it doesn't matter which values we choose, so we might as well choose the easiest ones). Now use the result from Problem 3 to find explicit formulas for the three roots r_1, r_2, r_3.

For part (a), we are given that

$$\begin{aligned} x^3 - 6x - 6 &= (x - r_1)(x - r_2)(x - r_3) \\ &= x^3 - (r_1 + r_2 + r_3)x^2 + (r_1 r_2 + r_1 r_3 + r_2 r_3)x - (r_1 r_2 r_3) \\ &= x^3 - e_1 x^2 + e_2 x - e_3, \end{aligned}$$

and hence that $e_1 = 0$, $e_2 = -6$, and $e_3 = 6$. This implies that

$$s_2^3 + s_3^3 = 2e_1^3 - 9e_1 e_2 + 27e_3 = 2 \cdot 0^2 - 9 \cdot 0(-6) + 27 \cdot 6 = 162$$

and

$$s_2 s_3 = e_1^2 - 3e_2 = 0^2 - 3(-6) = 18.$$

Now e_2^3 and e_3^3 are the roots of the polynomial

$$\begin{aligned} (u - s_2^3)(u - s_3^3) &= u^2 - (s_2^3 + s_3^3)u + s_2^3 s_3^3 \\ &= u^2 - (s_2^3 + s_3^3)u + (s_2 s_3)^3 \\ &= u^2 - 162u + 18^2 \\ &= u^2 - 162u + 324. \end{aligned}$$

Using the quadratic formula gives

$$\begin{aligned} s_2^3, s_3^3 &= \frac{1}{2}(162 \pm \sqrt{162^2 - 4 \cdot 324^2}) \\ &= \frac{1}{2}(162 \pm 54) \\ &= 54, 108. \end{aligned}$$

It doesn't matter which is which, so let's just say $s_2^3 = 54$ and $s_3^3 = 108$.

For part (b) we take the real cube roots to get

$$s_2 = \sqrt[3]{54} = 3 \cdot \sqrt[3]{2} \qquad \text{and} \qquad s_3 = \sqrt[3]{108} = 3 \cdot \sqrt[3]{4}$$

Finally, we use the result of Problem 3 to compute the roots of $x^3 - 6x - 6$:

$$\begin{aligned} r_1 &= (s_1 + s_2 + s_3)/3 = \sqrt[3]{2} + \sqrt[3]{4} \\ r_2 &= (s_1 + \omega^2 s_2 + \omega s_3)/3 = \omega^2 \cdot \sqrt[3]{2} + \omega \cdot \sqrt[3]{4} \\ r_3 &= (s_1 + \omega s_2 + \omega^2 s_3)/3 = \omega \cdot \sqrt[3]{2} + \omega^2 \cdot \sqrt[3]{4}. \end{aligned}$$

If you don't believe it, I encourage you to check that these three numbers satisfy the equation $x^3 - 6x - 6 = 0$. Since a polynomial of degree 3 has at most 3 roots, these must be all of them. See the last page of the Course Notes for a picture of these roots in the complex plane.