Math 461 **Spring 2015**
Homework 4 **Drew Armstrong**

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1. Difference of Like Powers. Let n be a positive integer and define $\omega := e^{2\pi i/n}$. Prove that for all numbers a and b we have

$$a^n - b^n = (a-b)(a-\omega b)(a-\omega^2 b)\cdots(a-\omega^{n-1}b).$$

Proof 1: First note that the formula is true for $b = 0$:

$$a^n - 0 = (a-0)(a-0)(a-0)\cdots(a-0).$$

Next suppose that $b \neq 0$. We know that for all numbers x we have

$$x^n - 1 = (x-1)(x-\omega)(x-\omega^2)\cdots(x-\omega^{n-1}).$$

This is because of Descartes' Factor Theorem and the fact that $1, \omega, \ldots, \omega^{n-1}$ are the roots of the polynomial $x^n - 1$. Since $b \neq 0$ we can substitute $x = \frac{a}{b}$ to get

$$\begin{aligned} \frac{a^n}{b^n} - 1 &= \left(\frac{a}{b} - 1\right)\left(\frac{a}{b} - \omega\right)\left(\frac{a}{b} - \omega^2\right)\cdots\left(\frac{a}{b} - \omega^{n-1}\right) \\ \frac{a^n - b^n}{b^n} &= \frac{(a-b)}{b}\frac{(a-\omega b)}{b}\frac{(a-\omega^2 b)}{b}\cdots\frac{(a-\omega^{n-1})}{b}. \end{aligned}$$

Finally, multiply both sides by b^n to get the desired formula. ///

Proof 2: Alternatively, we can note that the nth roots of b^n are

$$b, \omega b, \omega^2 b, \ldots, \omega^{n-1}b.$$

Then by Descartes' Factor Theorem we can factor the polynomial $x^n - b^n$ as

$$x^n - b^n = (x-b)(x-\omega b)(x-\omega^2 b)\cdots(x-\omega^{n-1}b).$$

Now substitute $x = a$ to get the desired formula. ///

2. Roots of Numbers Other Than 1.

(a) Compute the fourth roots of -1.
(b) Use part (a) to factor $x^4 + 1$ over the **real numbers**.

For part (a) we want to solve the equation $x^4 = -1$. Let $x = re^{i\theta}$ in polar coordinates. Then we have

$$r^4 e^{i4\theta} = -1 = e^{i\pi}$$

which implies that $r = 1$ and

$$\begin{aligned} 4\theta - \pi &= 2\pi k \\ \theta &= \frac{2\pi k + \pi}{4} \\ \theta &= \frac{\pi}{4} + \frac{\pi}{2}k \end{aligned}$$

for any $k \in \mathbb{Z}$. This gives us four solutions:

$$x = e^{i\pi/4}, e^{i3\pi/4}, e^{i5\pi}4, e^{i7\pi}4.$$

In other words
$$x = \frac{1+i}{\sqrt{2}}, \frac{1-i}{\sqrt{2}}, \frac{-1+i}{\sqrt{2}}, \frac{-1-i}{\sqrt{2}}.$$

For part (b) we first note from part (a) that
$$x^4+1 = \left(x - \frac{1+i}{\sqrt{2}}\right)\left(x - \frac{1-i}{\sqrt{2}}\right)\left(x - \frac{-1+i}{\sqrt{2}}\right)\left(x - \frac{-1-i}{\sqrt{2}}\right)$$
Unfortunately these factors have complex coefficients. To get real coefficients we multiply them together in conjugate pairs:
$$\begin{aligned} x^4+1 &= \left[\left(x - \frac{1+i}{\sqrt{2}}\right)\left(x - \frac{1-i}{\sqrt{2}}\right)\right]\left[\left(x - \frac{-1+i}{\sqrt{2}}\right)\left(x - \frac{-1-i}{\sqrt{2}}\right)\right] \\ x^4+1 &= (x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1). \end{aligned}$$

[It is easy to check that this factorization is correct, but it is not so easy to come up with it unless you know about the polar form of complex numbers. In 1702 Gottfried Wilhelm Leibniz (one of the inventors of the Calculus) claimed that polynomials of the form x^4+a^4 are irreducible over $\mathbb{R}$. He was wrong, as you see.]

3. Cyclotomic Polynomials. We say that $\zeta \in \mathbb{C}$ is a **primitive** nth root of 1 if (1) $\zeta^n = 1$ and (2) $\zeta^m \neq 1$ for $m < n$. The nth cyclotomic polynomial is defined by
$$\Phi_n(x) := \prod_{\zeta}(x-\zeta)$$
where ζ runs over the primitive nth roots of 1.

(a) Find all the primitive 8th roots of 1.
(b) Use part (a) to compute $\Phi_8(x)$.
(c) Use part (b) to completely factor x^8-1 over the integers.

For part (a) let $\omega := e^{2\pi i/8} = e^{i\pi/4}$. The 8th roots of 1 are
$$1, \omega, \omega^2, \omega^3, \omega^4, \omega^5, \omega^6, \omega^7.$$
Among these, note that 1 is a first root of 1, 1 and ω^4 are second roots of 1, and $1, \omega^2, \omega^4, \omega^6$ are fourth roots of 1. Thus the primitive 8th roots of 1 are
$$\omega = e^{i\pi/4}, \ \ \omega^3 = e^{i3\pi/4}, \ \ \omega^5 = e^{i5\pi/4}, \ \ \omega^7 = e^{i7\pi/4}.$$
You may notice from Problem 2 that these are the same as the 4th roots of -1. [Did I plan that?]

For part (b) we can use part (a) and the results of Problem 2 to see that
$$\Phi_8(x) = (x-\omega)(x-\omega^3)(x-\omega^5)(x-\omega^7) = x^4+1.$$

For part (c) we have two options. First, we can recall the formula
$$x^8-1 = \Phi_1(x)\Phi_2(x)\Phi_4(x)\Phi_8(x).$$
We can recall (or compute) that $\Phi_1(x) = (x-1)$, $\Phi_2(x) = (x+1)$, and $\Phi_4(x) = x^2+1$, hence
$$x^8-1 = (x-1)(x+1)(x^2+1)(x^4+1).$$

Second, we can use the fact that $x^8 - 1$ is a difference of squares to write

$$x^8 - 1 = (x^4 + 1)(x^4 - 1)$$

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$$x^8 - 1 = (x^4 + 1)(x^2 + 1)(x^2 - 1)$$

and then use the fact that $x^2 - 1$ is a difference of squares to write

$$x^8 - 1 = (x^4 + 1)(x^2 + 1)(x + 1)(x - 1).$$

4. Trisecting an Angle.

(a) Use de Moivre's Theorem to express $\cos(3\theta)$ as a polynomial in $\cos(\theta)$.
(b) Solve the polynomial equation from part (a) to express $\cos(\theta)$ in terms of $\cos(3\theta)$.
(c) Use part (b) to find the exact value of $\cos(\pi/9)$.

For part (a) we first use de Moivre's Theorem to get

$$\begin{aligned}\cos(3\theta) + i\sin(3\theta) &= (\cos\theta + i\sin\theta)^3 \\ &= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta \\ &= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^3\theta\sin\theta - \sin^3\theta).\end{aligned}$$

Then by equating the real parts and using the fact that $\sin^2\theta = 1 - \cos^2\theta$ gives

$$\begin{aligned}\cos(3\theta) &= \cos^3\theta - 3\cos\theta\sin^2\theta \\ &= \cos^3\theta - 3\cos\theta(1 - \cos^2\theta) \\ &= 4\cos^3\theta - 3\cos\theta.\end{aligned}$$

For part (b) let $x = \cos\theta$ and $c = \cos(3\theta)$. We want to solve the following equation for x:

$$\begin{aligned}4x^3 - 3x &= c, \\ x^3 - \frac{3}{4}x - \frac{c}{4} &= 0.\end{aligned}$$

Luckily this is a depressed cubic so we can use Cardano's Formula with $p = -\frac{3}{4}$ and $q = -\frac{c}{4}$ to get

$$\begin{aligned}x &= \sqrt[3]{\frac{c}{8} + \sqrt{\left(-\frac{c}{8}\right)^2 + \left(-\frac{1}{4}\right)^3}} + \sqrt[3]{\frac{c}{8} - \sqrt{\left(-\frac{c}{8}\right)^2 + \left(-\frac{1}{4}\right)^3}} \\ &= \sqrt[3]{\frac{c}{8} + \sqrt{\frac{c^2 - 1}{64}}} + \sqrt[3]{\frac{c}{8} - \sqrt{\frac{c^2 - 1}{64}}}.\end{aligned}$$

We can simplify this a bit if we write $s = \sin(3\theta)$ so that $c^2 - 1 = -s^2$. Then we have

$$\begin{aligned}x &= \sqrt[3]{\frac{c}{8} + \sqrt{\frac{-s^2}{64}}} + \sqrt[3]{\frac{c}{8} - \sqrt{\frac{-s^2}{64}}} \\ &= \sqrt[3]{\frac{c}{8} + \frac{is}{8}} + \sqrt[3]{\frac{c}{8} - \frac{is}{8}} \\ &= \frac{1}{2}\sqrt[3]{c + is} + \frac{1}{2}\sqrt[3]{c - is}.\end{aligned}$$

In other words,

$$\cos\theta = \frac{1}{2}\sqrt[3]{\cos(3\theta) + i\sin(3\theta)} + \frac{1}{2}\sqrt[3]{\cos(3\theta) - i\sin(3\theta)}.$$

That's not a "nice" formula, but it is a formula. ///

For part (c) we use our not-nice formula to find the exact value of $\cos(\pi/9)$. We put $\theta = \pi/9$ in the formula to get

$$\begin{aligned}\cos(\pi/9) &= \frac{1}{2}\sqrt[3]{\cos(\pi/3) + i\sin(\pi/3)} + \frac{1}{2}\sqrt[3]{\cos(\pi/3) - i\sin(\pi/3)} \\ &= \frac{1}{2}\sqrt[3]{\frac{1+i\sqrt{3}}{2}} + \frac{1}{2}\sqrt[3]{\frac{1-i\sqrt{3}}{2}}.\end{aligned}$$

[We might wonder if this not-nice formula for $\cos(\pi/9)$ can be simplified. We will soon prove in class that the answer is NO IT CANNOT. Specifically, we will prove that the number $\cos(\pi/9)$ is not constructible, i.e., it cannot be expressed via the integers using only field operations and square roots.]

5. Rational Root Test. Let $f(x) \in \mathbb{Z}[x]$ be a polynomial with integer coefficients, say $f(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n$ with $c_n \neq 0$.

(a) If $f(a/b) = 0$ for some integers $a, b \in \mathbb{Z}$ with no common factor, prove that a divides c_0 and b divides c_n. [Hint: Multiply both sides of $f(a/b) = 0$ by b^n.]

(b) Use part (a) to prove that the polynomial $f(x) = x^3 - 3x - 1$ has **no rational root**.

For part (a) we assume that $a, b \in \mathbb{Z}$ have no common factor. This means we can assume $b \neq 0$ (since otherwise a would be a common factor). Now suppose that $f(a/b) = 0$, i.e.,

$$c_0 + c_1\frac{a}{b} + c_2\frac{a^2}{b^2} + \cdots + c_n\frac{a^n}{b^n} = 0.$$

Multiply both sides by b^n to get

$$\begin{aligned}c_0b^n + c_1ab^{n-1} + c_2a^2b^{n-2} + \cdots + c_na^n &= 0 \\ c_1ab^{n-1} + c_2a^2b^{n-2} + \cdots + c_na^n &= -c_0b^n \\ a(c_1b^{n-1} + c_2ab^{n-1} + \cdots c_na^{n-1}) &= -c_0b^n.\end{aligned}$$

We conclude that a divides the number c_0b^n. Since a and b have no common factor, this implies that a divides the number c_0, as desired. The proof that b divides c_n is similar.

For part (b) consider $f(x) = x^3 - 3x - 1 \in \mathbb{Z}[x]$. Assume for contradiction that we have $f(a/b) = 0$ for some $a, b \in \mathbb{Z}$. Then part (a) tells us that a divides -1 (hence $a = \pm 1$) and b divides 1 (hence $b = \pm 1$), hence $a/b = \pm 1$. But notice that

$$f(1) = 1 - 3 - 1 = -3 \neq 0 \qquad \text{and} \qquad f(-1) = -1 + 3 - 1 = 1 \neq 0.$$

Contradiction. ///

[We will see later that Problem 5(b) is the final step in the proof that $\cos(\pi/9)$ is not a constructible number (and hence it is impossible to trisect the angle with straightedge and compass). Stay tuned for the rest of the proof.]