Math 461 **Spring 2015**
Homework 3 **Drew Armstrong**

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1. De Moivre's Theorem.

(a) Use de Moivre's Theorem to express $\cos(2\theta)$ as a polynomial in $\cos(\theta)$.
(b) Solve this polynomial to obtain a formula for $\cos(\theta)$ in terms of $\cos(2\theta)$.
(c) Use the formula from (b) to find the exact value of $\cos(\pi/8)$.

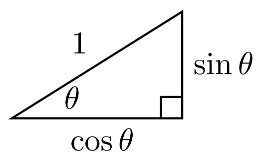
For part (a) we have

$$\begin{aligned}\cos(2\theta) + i\sin(2\theta) &= (\cos\theta + i\sin\theta)^2 \\ &= \cos\theta\cos\theta + 2i\sin\theta\cos\theta + i^2\sin\theta\sin\theta \\ &= (\cos^2\theta - \sin^2\theta) + i(2\sin\theta\cos\theta).\end{aligned}$$

Comparing the real parts of both sides gives

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta.$$

Now remember that $\cos^2\theta + \sin^2\theta = 1$. Why? This is just the Pythagorean Theorem:



Thus we have

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = \cos^2\theta - (1 - \cos^2\theta) = 2\cos^2\theta - 1. \tag{1}$$

For part (b), we solve equation (1) to obtain

$$\cos\theta = \sqrt{\frac{\cos(2\theta) + 1}{2}}.$$

This is sometimes called the "half-angle formula" because it allows us to compute $\cos(\theta/2)$ whenever we know $\cos\theta$:

$$\cos(\theta/2) = \sqrt{\frac{\cos\theta + 1}{2}} = \frac{1}{2}\sqrt{2 + 2\cos\theta}.$$

For part (c), start with an angle you know. Do you know that $\cos(\pi/2) = 0$? Good. Then

$$\cos(\pi/4) = \frac{1}{2}\sqrt{2 + 2\cos(\pi/2)} = \frac{1}{2}\sqrt{2 + 2\cdot 0} = \frac{1}{2}\sqrt{2}.$$

Applying the formula again gives

$$\cos(\pi/8) = \frac{1}{2}\sqrt{2 + 2\cos(\pi/4)} = \frac{1}{2}\sqrt{2 + \sqrt{2}}.$$

Just for fun, let's also compute $\cos(\pi/16)$:

$$\cos(\pi/16) = \frac{1}{2}\sqrt{2 + 2\cos(\pi/8)} = \frac{1}{2}\sqrt{2 + \sqrt{2 + \sqrt{2}}}.$$

Hey, now I see a pattern. And it tells me that

$$\lim_{n\to\infty} \cos\left(\frac{\pi}{2^n}\right) = \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}}.$$

But of course we know that

$$\lim_{n\to\infty} \cos\left(\frac{\pi}{2^n}\right) = \cos(0) = 1,$$

hence

$$2 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}}.$$

That's *was* kind of fun, right?

2. Quadratic Formula Again.

(a) Compute the square roots of i.
(b) Use part (a) to solve the equation $\frac{1}{2}z^2 + (1+i)z + \frac{i}{2} = 0$ for $z \in \mathbb{C}$.

For part (a) we want to solve the equation $x^2 = i$. To do this we will express x and i in polar coordinates: let $x = re^{i\theta}$ and note that $i = e^{i\pi/2}$. Then we have

$$\begin{aligned} x^2 &= i \\ (re^{i\theta})^2 &= e^{i\pi/2} \\ r^2 e^{i2\theta} &= e^{i\pi/2} \end{aligned}$$

Comparing lengths gives $r = 1$ and comparing angles gives

$$\begin{aligned} 2\theta - \pi/2 &= 2\pi k \\ 2\theta &= 2\pi k + \pi/2 \\ \theta &= \frac{2\pi k + \pi/2}{2} \\ \theta &= \pi/4 + \pi k \end{aligned}$$

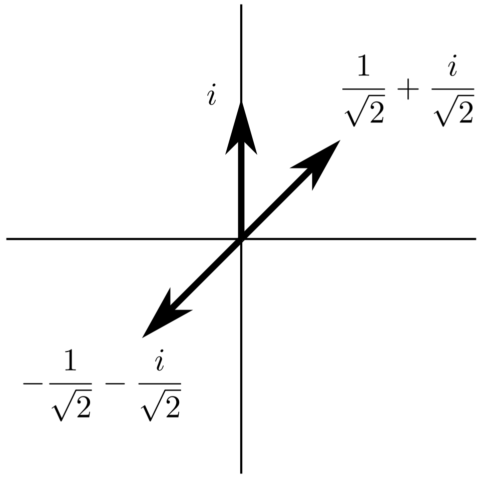
for any $k \in \mathbb{Z}$. We conclude that the square roots of i are

$$x = \sqrt{i} = e^{\pi/4+\pi k}, \text{ for any } k \in \mathbb{Z}.$$

That looks like a lot, but it really just represents **two** complex numbers:

$$e^{i\pi/4} = (1+i)/\sqrt{2} \qquad \text{and} \qquad e^{i(\pi/4+\pi)} = -(1+i)/\sqrt{2}.$$

Here is a picture of i and its two square roots:



For part (b) we apply the good old Quadratic Formula to get

$$z = -(1+i) + \sqrt{(1+i)^2 - i} = -(1+i) + \sqrt{(1+2i-1) - i} = -(1+i) + \sqrt{i}.$$

Finally, we use the square roots of i computed in part (a) to get

$$z = -(1+i) + \frac{1}{\sqrt{2}}(1+i) = \left(-1 + \frac{1}{\sqrt{2}}\right)(1+i) = \left(\frac{-2+\sqrt{2}}{2}\right)(1+i),$$

or

$$z = -(1+i) - \frac{1}{\sqrt{2}}(1+i) = \left(-1 - \frac{1}{\sqrt{2}}\right)(1+i) = \left(\frac{-2-\sqrt{2}}{2}\right)(1+i),$$

3. Complex Conjugation. Recall that complex conjgation $* : \mathbb{C} \to \mathbb{C}$ is defined by

$$(a+ib)^* := a - ib.$$

Show that for all $u, v \in \mathbb{C}$ we have

(a) $(u+v)^* = u^* + v^*$
(b) $(uv)^* = u^*v^*$
(c) $|u||v| = |uv|$. [Hint: $|u|^2 = uu^*$.]

Let $u = a + ib$ and $v = c + id$ where a, b, c, d are real. For part (a) we have

$$\begin{aligned} u^* + v^* &= (a+ib)^* + (c+id)^* \\ &= (a-ib) + (c-id) \\ &= (a+c) - i(b+d) \\ &= ((a+c) + i(b+d))^* \\ &= ((a+ib) + (c+id))^* \\ &= (u+v)^*. \end{aligned}$$

For part (b) we have

$$\begin{aligned}
u^*v^* &= (a+ib)^*(c+id)^* \\
&= (a-ib)(c-id) \\
&= (ac-bd)+i(-ad-bc) \\
&= (ac-bd)-i(ad+bc) \\
&= ((ac-bd)+i(ad+bc))^* \\
&= ((a+ib)(c+id))^* \\
&= (uv)^*.
\end{aligned}$$

For part (c) we don't need to do any more real work, because it follows directly from (b) that

$$\begin{aligned}
|u|^2|v|^2 &= (uu^*)(vv^*) \\
&= (uv)(u^*v^*) \\
&= (uv)(uv)^* \\
&= |uv|^2.
\end{aligned}$$

Now take the positive square root of both sides.

4. Conjugate Pairs of Roots.

(a) Consider a polynomial with **real** coefficients, $f(x) \in \mathbb{R}[x]$. Show that for all **complex** numbers $z \in \mathbb{C}$ we have $f(z)^* = f(z^*)$.

(b) Conclude that the **complex** roots of a **real** polynomial come in conjugate pairs.

For part (a), assume that $f(x) = \sum_{k\geq 0} a_k x^k$ is a polynomial with **real** coefficients, and let z be any **complex** number. Then $f(z)$ is a complex number so we can compute its conjugate. Since the coefficients a_k are real we have $(a_k)^* = a_k$ for all k. Using Problem 3(a) and then 3(b) gives

$$\begin{aligned}
f(z)^* &= \left(\sum_{k\geq 0} a_k z^k\right)^* \\
&= \sum_{k\geq 0} (a_k z^k)^* && 3(a) \\
&= \sum_{k\geq 0} (a_k)^* (z^k)^* && 3(b) \\
&= \sum_{k\geq 0} a_k (z^k)^* && a_k \text{ is real} \\
&= \sum_{k\geq 0} a_k (z^*)^k && ? \\
&= f(z^*).
\end{aligned}$$

In the last step we used the fact that $(z^k)^* = (z^*)^k$. Why is this true? Because of 3(b):

$$(z^k)^* = (\underbrace{z \cdot z \cdots z}_{k \text{ times}})^* = \underbrace{z^* \cdot z^* \cdots z^*}_{k \text{ times}} = (z^*)^k.$$

Technically, we should use induction on k to prove this, but why bother?

For part (b), assume that $f(x)$ is a polynomial with **real** coefficients and suppose that $f(z) = 0$ for some **complex** number z. Then from part (a) we have

$$f(z^*) = f(z)^* = 0^* = 0,$$

hence z^* is also a root. This implies that the complex (non-real) roots of $f(x)$ come in complex-conjugate pairs. In other words, the set of roots of $f(x)$ in the complex plane has a reflection symmetry across the real axis.

5. Useful Little Theorem. Let $f(x)$ be a polynomial of degree 3 with **real** coefficients. Prove that if $f(x)$ has a **complex** root, then it must also have a **real** root. [Hint: If $f(u) = 0$ for some $u \in \mathbb{C}$, show that $f(x)$ is divisible by $(x^2 - (u + u^*)x + uu^*)$. Show that the quotient must have real coefficients.]

You might think that this is a Useless Little Theorem, because we already know (by the Intermediate Value Theorem) that every real polynomial of degree 3 has a real root. That's a valid objection, but we will see later that this theorem is surprisingly useful in a different context. For example, we will use it to prove that a regular 7-gon can not be constructed with a ruler and compass.

Proof. Suppose that $f(x) \in \mathbb{R}[x]$ has degree 3 and suppose that $f(u) = 0$ for some $u \in \mathbb{C}$. We will show that $f(x)$ has a real root. If u is real then we're done. Otherwise, Problem 4(b) implies that u^* is another root of $f(x)$. Using Descartes' Factor Theorem gives

$$f(x) = (x - u)(x - u^*)g(x)$$

where $g(x)$ is a polynomial of degree 1, say $g(x) = ax + b$. But then $g(-b/a) = 0$ and hence $f(-b/a) = 0$. We will be done if we can show that $-b/a$ is a **real** number. In fact, we will show that a and b are both real.

This might seem obvious, but it's not. We know that $f(x)$ has real coefficients, but there are complex numbers on the right side of the equation and it's possible that they cancel in some complicated way. The key is to expand

$$(x - u)(x - u^*) = (x^2 - (u + u^*)x + uu^*),$$

and to observe that $u+u^*$ and uu^* are both **real**. [Why is this?] Thus $g(x)$ is the quotient of a real polynomial divided by a real polynomial. This implies that $g(x)$ has real coefficients. □

[That last part (showing that $g(x)$ has real coefficients) was a little bit subtle. I expect many people got confused, and that's OK! As I mentioned before, it is not immediately obvious that this Useful Little Theorem is interesting. You'll have to take my word for it and be patient.]