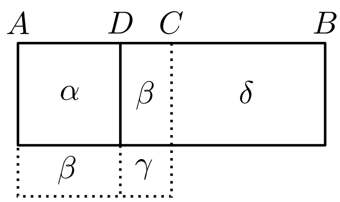
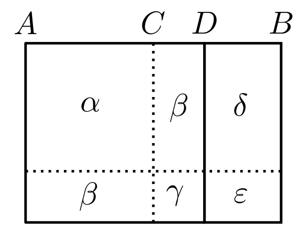
Math 461 **Spring 2015**
Homework 1 Solutions **Drew Armstrong**

1. In al-Khwarizmi's solution of quadratic equations he needed to solve the following geometric problem. Consider a line segment AB. Let C be its midpoint and let D be any other point on the segment. Construct a square on AD and complete this to a rectangle on AB. There are two different ways this could look (see the solid lines):





In both cases give a geometric argument that the area of the solid rectangle on DB plus the area of the square on CD equals the area of the square on AC. [Hint: Divide the diagrams by the suggested dotted lines. The Greek letters represent different areas in the two diagrams.]

First we consider the diagram on the left. We are asked to show that

$$(\beta + \delta) + (\gamma) = (\alpha + \beta + \beta + \gamma).$$

Indeed, since C is the midpoint of AB we note that area $\alpha + \beta$ equals area δ, because they are both half of the solid rectangle on AB. Hence

$$\begin{aligned}(\alpha + \beta + \beta + \gamma) &= (\alpha + \beta) + \beta + \delta \\ &= \delta + \beta + \gamma \\ &= (\beta + \delta) + (\gamma).\end{aligned}$$

Next we consider the diagram on the right. We are asked to show that

$$(\delta + \varepsilon) + (\gamma) = (\alpha).$$

Indeed, since C is the midpoint of AB we note that area $\alpha + \beta$ equals area $\beta + \gamma + \delta + \varepsilon$, because they are both half of the solid rectangle on AB. Hence

$$\begin{aligned}\alpha + \beta &= \beta + \gamma + \delta + \varepsilon \\ \alpha &= \gamma + \delta + \varepsilon \\ (\alpha) &= (\gamma) + (\delta + \varepsilon).\end{aligned}$$

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2. Consider the quadratic equation $(x - r)(x - s) = 0$, where r and s are constants.

(a) Show that the discriminant of this equation is $(r - s)^2$.
(b) Show that the discriminant is zero if and only if $r = s$.

(a) First we expand the polynomial to obtain

$$\begin{aligned}(x-r)(x-s) &= 0\\ x^2 - rx - sx + rs &= 0\\ x^2 - (r+s)x + (rs) &= 0.\end{aligned}$$

Thus the discriminant is

$$\begin{aligned}(-(r+s))^2 - 4\cdot 1\cdot (rs) &= (r+s)^2 - 4rs\\ &= r^2 + 2rs + s^2 - 4rs\\ &= r^2 - 2rs + s^2\\ &= (r-s)^2.\end{aligned}$$

(b) If $r = s$ then we have $r - s = 0$ and hence $(r-s)^2 = 0^2 = 0$. Conversely, suppose that $(r-s)^2 = (r-s)(r-s) = 0$. This implies that either $(r-s) = 0$ or $(r-s) = 0$. In either case, we have $r - s = 0$, and hence $r = s$. ///

[Remark: We call r and s the roots of the equation (and this is why I chose the letter "r"). We have just shown that the discriminant of a quadratic is zero if and only if the two roots are equal. In the past we have seen quadratics with negative discriminant. How could the number $(r-s)^2$ ever be negative?]

3. Suppose that the quadratic equation $x^2+px+q=0$ has solutions $x = r$ and $x = s$. Find a quadratic equation with solutions $x = 1/r$ and $x = 1/s$. [Hint: Use $(x-r)(x-s) = x^2+px+q$ to express p and q in terms of r and s. Now consider $(x-1/r)(x-1/s)$.]

Suppose the equation $x^2 + px + q = 0$ has solutions $x = r$ and $x = s$. Then by Descartes' Factor Theorem we know that

$$x^2 + px + q = (x-r)(x-s) = x^2 - (r+s)x + rs.$$

From this it follows that $p = -(r+s)$ and $q = rs$. [Why?] Now we wish to find a quadratic equation with solutions $x = 1/r$ and $x = 1/s$. The most obvious such equation is

$$(x-1/r)(x-1/s) = 0.$$

To find the coefficients of this equation we expand:

$$\begin{aligned}(x-1/r)(x-1/s) &= x^2 - \left(\frac{1}{r} + \frac{1}{s}\right)x + \frac{1}{rs}\\ &= x^2 - \left(\frac{r+s}{rs}\right)x + \frac{1}{rs}\\ &= x^2 + \frac{p}{q}x + \frac{1}{q}.\end{aligned}$$

Thus our equation has the form

$$\begin{aligned}x^2 + \frac{p}{q}x + \frac{1}{q} &= 0\\ qx^2 + px + 1 &= 0.\end{aligned}$$

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[Remark: Note that we just reversed the coefficients of the original polynomial. Try to show that reversing the coefficients is always the same as inverting the roots of a polynomial equation.]

4. Factor the following cubic polynomials as $f(x) = (x-r)(x-s)(x-t)$ by: (1) guessing a solution to $f(x) = 0$, (2) using long division, (3) using the quadratic formula.

(a) $f(x) = x^3 - 3x^2 + x + 1$
(b) $f(x) = x^3 - 1$

(a) First we observe that $f(1) = 1 - 3 + 1 + 1 = 0$. Next we divide $f(x)$ by $(x-1)$ to get

$$
\begin{array}{r|l}
 & x^2 - 2x - 1 \\
\hline
x - 1 & x^3 - 3x^2 + x + 1 \\
 & -x^3 + x^2 \\
\hline
 & -2x^2 + x \\
 & 2x^2 - 2x \\
\hline
 & -x + 1 \\
 & x - 1 \\
\hline
 & 0
\end{array}
$$

The remainder is zero, as guaranteed by Descartes' Factor Theorem. Now we have $f(x) = (x-1)x^2 - 2x - 1$. In order to factor $x^2 - 2x - 1$ we apply the Quadratic Formula. The equation $x^2 - 2x - 1 = 0$ has solutions

$$x = \frac{2 \pm \sqrt{8}}{2} = \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}.$$

(Here I use $\sqrt{2}$ to represent the positive square root of 2.) Descartes' Factor Theorem now tells us that

$$x^2 - 2x - 1 = (x - (1 + \sqrt{2}))(x - (1 - \sqrt{2})) = (x - 1 - \sqrt{2})(x - 1 + \sqrt{2}).$$

In conclusion, we have

$$x^3 - 3x^2 + x + 1 = (x-1)(x - 1 - \sqrt{2})(x - 1 + \sqrt{2}).$$

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(b) First we observe that $f(1) = 1 - 1 = 0$. Next we divide $f(x)$ by $(x-1)$ to obtain

$$
\begin{array}{r|l}
 & x^2 + x + 1 \\
\hline
x - 1 & x^3 \qquad\quad - 1 \\
 & -x^3 + x^2 \\
\hline
 & x^2 \\
 & -x^2 + x \\
\hline
 & x - 1 \\
 & -x + 1 \\
\hline
 & 0
\end{array}
$$

The remainder is zero, as guaranteed by Descartes' Factor Theorem. Now we have $f(x) = (x-1)(x^2 + x + 1)$. In order to factor $x^2 + x + 1$ we apply the Quadratic Formula. The equation $x^2 + x + 1 = 0$ has solutions

$$x = \frac{-1 \pm \sqrt{-3}}{2},$$

which implies that

$$x^2 + x + 1 = \left(x - \frac{-1 + \sqrt{-3}}{2}\right)\left(x - \frac{-1 - \sqrt{-3}}{2}\right).$$

(Here I use $\sqrt{-3}$ to represent one of the two square roots of -3. I don't care which one, and I don't care if this even makes sense. You may check that the algebra works out in any case.) In conclusion, we have

$$x^3 - 1 = (x-1)\left(x - \frac{-1+\sqrt{-3}}{2}\right)\left(x - \frac{-1-\sqrt{-3}}{2}\right).$$

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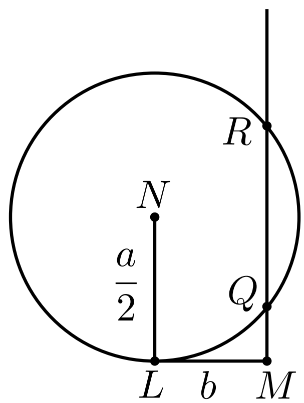
[Remark: That last factorization is certainly a true algebraic statement. However, it is less clear what meaning we should attach to the symbol $\sqrt{-3}$.]

5. Consider the following diagram from Descartes' *La Géométrie* (1637). Prove that the distances MQ and MR are solutions to the quadratic equation $y^2 + b^2 = ay$.



There are various geometric ways to do this. The easiest way is to consider point M as the origin $(0,0)$ of a Cartesian plane. Recall that the equation of a circle with radius ρ and center (α, β) is

$$(x-\alpha)^2 + (y-\beta)^2 = \rho^2.$$

Our circle has center $(-b, a/2)$ and radius $a/2$, so it has equation

$$(x+b)^2 + (y-a/2)^2 = (a/2)^2.$$

The equation of the line connecting Q and R is just $x = 0$. To compute the intersection of the line and circle we substitute $x = 0$ into the equation of the circle to get

$$\begin{aligned}(0-b)^2 + (y-a/2)^2 &= (a/2)^2\\ b^2 + y^2 - ay + (a/2)^2 &= (a/2)^2\\ b^2 + y^2 - ay &= 0\\ y^2 + b^2 &= ay.\end{aligned}$$

The solutions of this equation are the y-coordinates of the points Q and R, i.e., their distances from the origin M. ///

[Remark: The solutions of $y^2 + b^2 = ay$ are $y = (-a \pm \sqrt{a^2 - 4b^2})/2$. If the discriminant $a^2 - 4b^2$ is ≥ 0, then we can visualize this solution in terms of the points of intersection of the circle and line. If $a^2 - 4b^2 < 0$ then the line and circle don't intersect. Or do they?]