Math 461 **Spring 2015**
Exam 1 (Wed Feb 18) **Drew Armstrong**

There are 4 problems, each with 3 parts. Each part is worth 2 points, for a total of 24 points. If any two exams are submitted with copied answers then **both** exams will receive 0 points.

1. Division With Remainder.

(a) Let $\mathbb{F}$ be a field. Correctly state the Division Theorem for the ring $\mathbb{F}[x]$.

Given any polynomials $f(x), g(x) \in \mathbb{F}[x]$ with $g(x) \neq 0$, there exist unique polynomials $q(x), r(x) \in \mathbb{F}[x]$ such that

- $f(x) = q(x)g(x) + r(x)$, and
- $r(x) = 0$ or $\deg(r) < \deg(g)$.

(b) Divide $(x^3 - 4x^2 + 2x + 4)$ by $(x - 2)$ using long division.

$$\begin{array}{r|l} & x^2 - 2x - 2 \\ \hline x - 2 & x^3 - 4x^2 + 2x + 4 \\ & -x^3 + 2x^2 \\ \hline & -2x^2 + 2x \\ & 2x^2 - 4x \\ \hline & -2x + 4 \\ & 2x - 4 \\ \hline & 0 \end{array}$$

(c) Use the result of (b) to express $(x^3 - 4x^2 + 2x + 4)$ in the form $(x - r)(x - s)(x - t)$ for some r, s, t.

Part (b) tells us that $(x^3 - 4x^2 + 2x + 4) = (x - 2)(x^2 - 2x - 2)$. In order to factor $(x^2 - 2x - 2)$ we used the Quadratic Formula to solve $x^2 - 2x - 2 = 0$:

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(-2)}}{2} = \frac{2 \pm \sqrt{12}}{2} = \frac{2 \pm 2\sqrt{3}}{2} = 1 \pm \sqrt{3}.$$

We conclude that

$$(x^3 - 4x^2 + 2x + 4) = (x - 2)(x - 1 - \sqrt{3})(x - 1 + \sqrt{3}).$$

2. Multiplying Polynomials. Consider the two polynomials

$$f(x) = \sum_{k \geq 0} a_k x^k \quad \text{and} \quad g(x) = \sum_{k \geq 0} b_k x^k.$$

(a) Write a formula for the coefficient of x^k in the product $f(x)g(x)$.

$$a_0 b_k + a_1 b_{k-1} + \cdots + a_{k-1} b_1 + a_k b_0 \quad \text{or} \quad \sum_{i+j=k} a_i b_j \quad \text{or} \quad \sum_{i=0}^{k} a_i b_{k-i}.$$

(b) If $i+j > m+n$, prove that either $i > m$ or $j > n$ (or both).

Recall that the **contrapositive** of "if P then Q" is the statement "if not Q then not P", and that these two statements are logically equivalent. In our case $P =$ "$i+j > m+n$" and $Q =$ "$i > m$ or $j > n$". We will prove the contrapositive: Suppose that Q is not true, i.e., suppose that $i \leq m$ and $j \leq n$. If then follows that $i + j \leq m + n$, which implies that P is not true. Done. ///

(c) Now assume that $a_i = 0$ for all $i > m$ and $b_j = 0$ for all $j > n$. Use parts (a) and (b) to prove that the coefficient of x^k in $f(x)g(x)$ is zero for all $k > m + n$.

Assume that $a_i = 0$ for all $i > m$ and $b_j = 0$ for all $j > n$. If $i+j > m+n$ then by (b) we have $a_i = 0$ or $b_j = 0$, and hence $a_i b_j = 0$. The coefficient of x^k in $f(x)g(x)$ is, by part (a),

$$\sum_{i+j=k} a_i b_j.$$

If $k > m + n$, then each term in this sum is zero.

3. Cardano's Formula.

(a) What change of variables will convert the general cubic equation $ax^3+bx^2+cx+d = 0$ into a **depressed** cubic equation?

Substitute $x = y - b/3a$.

(b) Tell me Cardano's Formula for the solution of the depressed cubic $x^3 + px + q = 0$.

$$x = \sqrt[3]{-\left(\frac{q}{2}\right) + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{-\left(\frac{q}{2}\right) - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$$

(c) Use Cardano's Formula to find a solution to the equation $x^3 - 6x - 6 = 0$.

In this case we have $q/2 = -3$ and $p/3 = -2$, hence

$$\begin{aligned} x &= \sqrt[3]{-\left(\frac{q}{2}\right) + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{-\left(\frac{q}{2}\right) - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} \\ &= \sqrt[3]{3 + \sqrt{(-3)^2 + (-2)^3}} + \sqrt[3]{3 - \sqrt{(-3)^2 + (-2)^3}} \\ &= \sqrt[3]{3 + \sqrt{9-8}} + \sqrt[3]{3 - \sqrt{9-8}} \\ &= \sqrt[3]{4} + \sqrt[3]{2} \\ &\approx 2.847 \end{aligned}$$

4. Descartes' Theorem. Let $\mathbb{F}$ be a field and consider a polynomial $f(x) \in \mathbb{F}[x]$ of degree 2. Assume that $f(x)$ has two **distinct** roots $r, s \in \mathbb{F}$.

(a) Prove that $(x-r)$ divides $f(x)$ with remainder 0. Let $g(x)$ be the quotient.

By the Division Theorem (Problem 1(a)) we have

$$f(x) = (x-r)g(x) + R(x)$$

where either $R(x) = 0$ or $\deg(R) < \deg(x-r) = 1$. In either case, $R(x)$ must be a constant. Call it R. Then evaluate $f(x)$ at r to get

$$0 = f(r) = (r-r)g(r) + R = 0 \cdot g(r) + R = R.$$

We conclude that the remainder is zero.

(b) Explain why $g(s) = 0$.

From part (a) we have $f(x) = (x-r)g(x)$. Evaluating $f(x)$ at s gives

$$0 = f(s) = (s-r)g(s).$$

Since (by assumption) $s - r \neq 0$, we conclude that $g(s) = 0$.

(c) Prove that $g(x) = a(x-s)$ for some **nonzero** constant $a \in \mathbb{F}$.

Since $f(x) = (x-r)g(x)$ and $\deg(f) = 2$, we conclude that $\deg(g) = 1$. Then since $g(s) = 0$, we apply the same argument as in part (a) to show that

$$g(x) = (x-s)h(x)$$

for some polynomial $h(x)$. Comparing degrees again gives $\deg(h) = 0$. In other words, $h(x)$ is a nonzero constant. Call it a if you like.

[Many people assumed at the outset that $f(x) = a(x-r)(x-s)$, quoting a result from class. Since I guess I didn't **explicitly** tell you not to do this, these people received half points (minus any further mistakes).]

The average for this exam was 18.3/25 and the median was 18/25. Out of 40 students, 6 received a score of 23 or above. I do **not** assign letter grades for exams, but I estimate the following **approximate** grade ranges:

$$21 - 25 \approx A \text{ (13 students)}$$
$$17 - 20 \approx B \text{ (16 students)}$$
$$10 - 16 \approx C \text{ (11 students)}$$