Math 461 F — **Spring 2011**
Homework 2 Solutions — **Drew Armstrong**

A.1. Suppose that the cubic equation $ax^3 + bx^2 + cx + d = 0$ has three roots, called r, s, t. Give a formula for $rs + rt + st$ in terms of a, b, c, d.

By the Factor Theorem we can write

$$\begin{aligned} ax^3 + bx^2 + cx + d &= a(x-r)(x-s)(x-t) \\ &= ax^3 - a(r+s+t)x^2 + a(rs+rt+st)x - a(rst). \end{aligned}$$

Now recall that two polynomials are equal if and only if their coefficients are equal. Hence we have

$$rs + rt + st = \frac{c}{a}.$$

Note that $a \neq 0$ because it is the leading coefficient.

A.2. Find all complex solutions $z \in \mathbb{C}$ to the quadratic equation

$$z^2 - z + \left(\frac{1}{4} - \frac{i}{2}\right) = 0.$$

Note that

$$\begin{aligned} z^2 - z + \frac{1}{4} - \frac{1}{4} + \left(\frac{1}{4} - \frac{i}{2}\right) &= 0 \\ \left(z^2 - z + \frac{1}{4}\right) &= \frac{i}{2} \\ \left(z - \frac{1}{2}\right)^2 &= \frac{i}{2}. \end{aligned}$$

Hence $z^2 - 1/2$ must be a square root of $i/2$. There are two of these, and we can find them! Suppose that $x^2 = i/2$ with $x = r \operatorname{cis} \theta$ in polar form. Thus we have

$$x^2 = r^2 \operatorname{cis}(2\theta) = \frac{i}{2} = \frac{1}{2} \operatorname{cis}\left(\frac{\pi}{2}\right).$$

Since the lengths are equal we get $r^2 = 1/2$, or $r = 1/\sqrt{2}$. Since the angles are equal we get $2\theta = \pi/2 + 2\pi k$ for any integer k. In other words, $\theta = \pi/4$ or $\theta = 5\pi/4$. We conclude that the square roots of $i/2$ are

$$x = \frac{1}{\sqrt{2}} \operatorname{cis}\left(\frac{\pi}{4}\right) = \frac{1}{2} + \frac{i}{2} \quad \text{and} \quad x = \frac{1}{2} \operatorname{cis}\left(\frac{4\pi}{5}\right) = -\frac{1}{2} - \frac{i}{2}.$$

Finally, the solutions to the original equation are

$$z - \frac{1}{2} = \frac{1}{2} + \frac{i}{2} \quad \Rightarrow \quad z = 1 + \frac{i}{2}$$

and

$$z - \frac{1}{2} = -\frac{1}{2} - \frac{i}{2} \quad \Rightarrow \quad z = -\frac{i}{2}.$$

A.3. Use de Moivre's formula and the fact that $\cos^2\alpha + \sin^2\alpha = 1$ for all $\alpha \in \mathbb{R}$ to come up with a formula for $\cos(\theta/2)$ in terms of $\cos\theta$ alone. (You can assume $\cos(\theta/2) \geq 0$.) Use your formula to find the **exact** value of $\cos(\pi/8)$.

De Moivre's formula tells us that

$$\begin{aligned}\cos(2\alpha) + i\sin(2\alpha) &= (\cos\alpha + i\sin\alpha)^2 \\ &= (\cos^2\alpha - \sin^2\alpha) + i(2\sin\alpha\cos\alpha),\end{aligned}$$

and hence we get $\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$ for all $\alpha \in \mathbb{R}$. Then substituting $\alpha = \theta/2$ and using the Pythagorean Theorem yields

$$\begin{aligned}\cos\theta &= \cos^2(\theta/2) - \sin^2(\theta/2) \\ &= \cos^2(\theta/2) - (1 - \cos^2(\theta/2)) \\ &= 2\cos^2(\theta/2) - 1,\end{aligned}$$

and we may solve for $\cos(\theta/2)$ to get.

$$\cos\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+\cos\theta}{2}} = \sqrt{\frac{1}{2} + \frac{1}{2}\cos\theta}.$$

(Here we assume that $\theta/2$ is small — less than $\pi/2$ — so that $\cos(\theta/2)$ is a positive number.) To get a formula for $\cos(\pi/8)$, let's start with something that we know, like $\cos(\pi/2) = 0$. Then we use the formula repeatedly to get

$$\cos\left(\frac{\pi}{4}\right) = \sqrt{\frac{1}{2} + \frac{1}{2}\cos\left(\frac{\pi}{2}\right)} = \sqrt{\frac{1}{2} + \frac{1}{2}0} = \sqrt{\frac{1}{2}},$$

and

$$\cos\left(\frac{\pi}{8}\right) = \sqrt{\frac{1}{2} + \frac{1}{2}\cos\left(\frac{\pi}{4}\right)} = \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}},$$

and

$$\cos\left(\frac{\pi}{16}\right) = \sqrt{\frac{1}{2} + \frac{1}{2}\cos\left(\frac{\pi}{8}\right)} = \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}}.$$

Wait, I went too far.

A.4. Let $\omega = \cos(2\pi/3) + i\sin(2\pi/3)$. Prove that for any a, b we have

$$a^3 - b^3 = (a-b)(a-\omega b)(a-\omega^2 b).$$

Can you find a similar formula for the difference $a^n - b^n$ of nth powers? Hint: Factor $x^n - 1$ and then put $x = a/b$.

If $\omega = \text{cis}\,(2\pi/3)$, recall that $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$. Thus we have

$$\begin{aligned}
&(a-b)(a-\omega b)(a-\omega^2 b) \\
&= a^3 - a^2b(1+\omega+\omega^2) + ab^2\omega(1+\omega+\omega^2) - b^3\omega^3 \\
&= a^3 - b^3.
\end{aligned}$$

In general, I guess that

$$a^n - b^n = (a-b)(a-\omega b)(a-\omega^2 b)\cdots(a-\omega^{n-1}b),$$

where $\omega = \text{cis}(2\pi/n)$. Note that the above method of proof would be messy, so let's use a slicker way. First we use the Factor Theorem to write

$$x^n - 1 = (x-1)(x-\omega)(x-\omega^2)\cdots(x-\omega^{n-1}).$$

Since this formula holds for any x we can set $x = a/b$ to get

$$\left(\frac{a}{b}\right)^n - 1 = \left(\frac{a}{b} - 1\right)\left(\frac{a}{b} - \omega\right)\left(\frac{a}{b} - \omega^2\right)\cdots\left(\frac{a}{b} - \omega^{n-1}\right), \quad \text{or}$$

$$\frac{a^n - b^n}{b^n} = \frac{(a-b)}{b}\frac{(a-\omega b)}{b}\frac{(a-\omega^2 b)}{b}\cdots\frac{(a-\omega^{n-1}b)}{b}.$$

Now multiply both sides by b^n to get the result. (Note that this argument only works for $b \neq 0$. But if $b = 0$ then the formula still holds, so there's no problem.)

A.5. Prove that for every positive integer $n > 1$ we have

$$\sum_{k=1}^{n} \cos\frac{2\pi k}{n} = 0.$$

Hint: Consider the number $\omega = \cos(2\pi/n) + i\sin(2\pi/n)$.

First note that $\omega^k = \cos(2\pi k/n) + i\sin(2\pi k/n)$ by de Moivre's formula. We know that $\sum_{k=1}^{n} \omega^k = \omega \sum_{k=0}^{n-1} \omega^k = \omega \cdot 0 = 0$. Hence we can write

$$\begin{aligned}
0 + 0i &= \sum_{k=1}^{n} \omega^k \\
&= \sum_{k=1}^{n} \left[\cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right)\right] \\
&= \left[\sum_{k=1}^{n} \cos\left(\frac{2\pi k}{n}\right)\right] + i\left[\sum_{k=1}^{n} \sin\left(\frac{2\pi k}{n}\right)\right].
\end{aligned}$$

Equating the real parts of these two complex numbers gives the result.

A.6. Define a function $f : \mathbb{C} \to \mathrm{M}_{2\times 2}(\mathbb{R})$ from the complex numbers to the 2×2 real matrices by setting

$$f(a+ib) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

For any complex numbers $z, w \in \mathbb{C}$ verify the following:

(a) $f(z+w) = f(z) + f(w)$,
(b) $f(zw) = f(z)f(w)$,
(c) $|z|^2 = \det f(z)$.

(The operations on the right hand sides of the equations are matrix addition, matrix multiplication, and matrix determinant.)

Let $z = a + ib$ and $w = c + id$, where $a, b, c, d \in \mathbb{R}$. To see part (a) note that

$$\begin{aligned} f(z+w) &= f((a+c) + i(b+d)) \\ &= \begin{pmatrix} (a+c) & -(b+d) \\ (b+d) & (a+c) \end{pmatrix} \\ &= \begin{pmatrix} a & -b \\ b & a \end{pmatrix} + \begin{pmatrix} c & -d \\ d & c \end{pmatrix} \\ &= f(z) + f(w). \end{aligned}$$

For part (b), we have

$$\begin{aligned} f(z)f(w) &= \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} c & -d \\ d & c \end{pmatrix} \\ &= \begin{pmatrix} (ad-bc) & -(ac+bd) \\ (ac+bd) & (ad-bc) \end{pmatrix} \\ &= f((ad-bc) + i(ac+bd)) \\ &= f((a+ib)(c+id)) \\ &= f(zw). \end{aligned}$$

Here comes part (c):

$$\begin{aligned} \det f(z) &= \det \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \\ &= aa - b(-b) \\ &= a^2 + b^2 \\ &= |z|^2. \end{aligned}$$

What was the point of this exercise? I have stressed in class that the matrix form is the **most natural way** to think about complex numbers. This is because there is no funny "i" symbol hanging around (what is that thing anyway?) and because the addition and multiplication in this setting are just addition and multiplication of matrices (which are natural operations — trust me). However, I never **proved** in class that addition and multiplication are preserved. You just did so. In fancy language, you proved that the map f is a homomorphism from the ring of complex numbers to the ring of real 2×2 matrices. Homo="same". Morphism="structure". There is still some white space left. Should I say more? No.