Math 461 F **Spring 2011**
Homework 1 Solutions **Drew Armstrong**

Book Problems.

Let r and s be the roots of the quadratic equation $ax^2 + bx + c = 0$. The Factor Theorem then implies that

$$ax^2 + bx + c = a(x - r)(x - s) = ax^2 - a(r + s)x + crs.$$

Recall that two polynomials are equal if and only if their coefficients coincide. Hence we get $r + s = -b/a$ and $rs = c/a$, which completes **Exercises 1.1.5 and 1.1.6**. Next, for **Exercise 1.1.8** we have

$$\frac{1}{r} + \frac{1}{s} = \frac{r+s}{rs} = \frac{-b}{a} \cdot \frac{a}{c} = \frac{-b}{c}.$$

Exercise 1.1.16. Now let $r, s \neq 0$ be the roots of the quadratic $x^2 + px + q$, so that $r + s = -p$ and $rs = q$. In this case, what is "the" quadratic equation with roots $1/r$ and $1/s$? (Recall that the equation is unique up to a constant multiple.) By the Factor Theorem, the equation is

$$\left(x - \frac{1}{r}\right)\left(x - \frac{1}{s}\right) = 0,$$

and the left side simplifies to

$$\begin{aligned}\left(x - \frac{1}{r}\right)\left(x - \frac{1}{s}\right) &= x^2 - \left(\frac{1}{r} + \frac{1}{s}\right)x + \frac{1}{rs}\\ &= x^2 - \left(\frac{r+s}{rs}\right)x + \frac{1}{rs}\\ &= x^2 + \frac{p}{q}x + \frac{1}{q}.\end{aligned}$$

Thus the simplest way to write the equation is:

$$\boxed{qx^2 + px + 1 = 0.}$$

Exercise 1.1.17. For which real values of α are the roots of the equation $x^2 + \alpha x + \alpha = 0$ real? Answer: The quadratic formula gives the roots as

$$x = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\alpha}}{2},$$

and these will be real precisely when $\alpha^2 - 4\alpha \geq 0$. To solve this polynomial inequality, first factor to get $\alpha(\alpha - 4) \geq 0$. There are two ways that the product of real numbers α and $\alpha - 4$ can be nonnegative. Either: Both numbers are nonnegative, in which case we have $\alpha \geq 0$ and $\alpha - 4 \geq 0$ (i.e.

$\alpha \geq 4$). Or: Both numbers are nonpositive, in which case we have $\alpha \leq 0$ and $\alpha - 4 \leq 0$ (i.e. $\alpha \leq 4$). Conclusion: The roots will be real when we have

$$\boxed{\alpha \leq 0 \quad \text{or} \quad \alpha \geq 4.}$$

Additional Problems.

A.1. We will solve **Exercise 1.1.1** using the given hint. First note that

$$\begin{aligned}(\sqrt{3}+1)^3 &= (\sqrt{3})^3 + 3(\sqrt{3})^2 + 3(\sqrt{3}) + 1 \\ &= 3\sqrt{3} + 9 + 3\sqrt{3} + 1 \\ &= 6\sqrt{3} + 10 \\ &= \sqrt{108} + 10,\end{aligned}$$

and, similarly, that $(\sqrt{3}-1)^3 = \sqrt{108} - 10$. Finally, we have

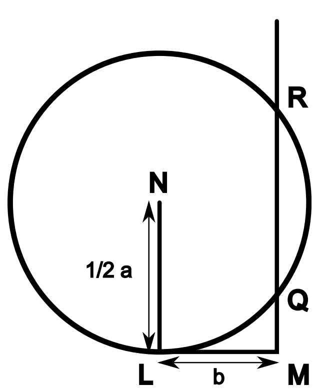
$$\sqrt[3]{\sqrt{108}+10} - \sqrt[3]{\sqrt{108}-10} = (\sqrt{3}+1) - (\sqrt{3}-1) = 2.$$

A.2. If the quadratic x^2+px+q has roots r and s, then the quantity $(r-s)^2$ is called the discriminant. We know that $r+s=-p$ and $rs=q$. Hence

$$(r-s)^2 = (r+s)^2 - 4rs = p^2 - 4q.$$

(Recall: Newton's theorem tells us that **any** symmetric function in r and s can be expressed in terms of the sum $r+s$ and product rs, although it may take a bit of work to do this.) Now note that r and s will be equal if and only if $(r-s)^2 = 0$. In other words, if and only if $p^2 - 4q = 0$.

A.3. Consider the following diagram from Descartes' *La Géométrie* (1637). Prove that the distances MQ and MR are solutions to the quadratic equation $y^2 = ay - b^2$.



If we place the point M at the origin of a Cartesian (x, y)-plane, then the circle has center $(-b, a/2)$ and radius $a/2$. **Recall** (I know you all know this) that the equation of a circle with radius R and center (α, β) is

$$(x-\alpha)^2 + (y-\beta)^2 = R^2.$$

Hence the equation of our circle is

$$(x+b)^2 + (y-a/2)^2 = (a/2)^2.$$

Now, the two distances MQ and MR are just the y-coordinates of the points of intersection of the circle with the y-axis. The equation of the y-axis is $x = 0$, and thus the points of intersection satisfy

$$\begin{aligned} (0+b)^2 + (y+a/2)^2 &= (a/2)^2 \\ b^2 + y^2 - ay + (a/2)^2 &= (a/2)^2 \\ b^2 + y^2 - ay &= 0 \\ y^2 &= ay - b^2, \end{aligned}$$

as desired.