Math 311 **Spring 2023**
Homework 6 **Drew Armstrong**

Given a square matrix A, a constant λ and a vector $\mathbf{v}$ satisfying

$$A\mathbf{v} = \lambda\mathbf{v},$$

we say that $\mathbf{v}$ *is an eigenvector of A with eigenvalue λ*. The eigenvalues of a given matrix are the roots of its *characteristic polynomial*. In the case of a 2×2 matrix we have

$$\begin{array}{ccc} \text{(matrix)} & \rightsquigarrow & \text{(characteristic polynomial)} \\ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \rightsquigarrow & \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0. \end{array}$$

Once an eigenvalue λ is found, the corresponding eigenvectors $\mathbf{v} = (u, v)$ can be found by solving a linear system:

$$A\mathbf{v} = \lambda\mathbf{v} \quad \rightsquigarrow \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \lambda\begin{pmatrix} u \\ v \end{pmatrix} \quad \rightsquigarrow \quad \begin{cases} (a-\lambda)u + bv &= 0, \\ cu + (d-\lambda)v &= 0. \end{cases}$$

Given the matrix A we may consider the system of first order linear differential equations:

$$\mathbf{x}'(t) = A\mathbf{x}(t) \quad \rightsquigarrow \quad \begin{cases} x'(t) &= ax(t) + by(t), \\ y'(t) &= cx(t) + dy(t). \end{cases}$$

If the matrix A has distinct eigenvalues $\lambda_1 \neq \lambda_2$ with corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ then the general solution of the system is

$$\mathbf{x}(t) = a_1 e^{\lambda_1 t}\mathbf{v}_1 + a_2 e^{\lambda_2 t}\mathbf{v}_2.$$

We may also consider the system of second order differential equations:

$$\mathbf{x}''(t) = A\mathbf{x}(t) \quad \rightsquigarrow \quad \begin{cases} x''(t) &= ax(t) + by(t), \\ y''(t) &= cx(t) + dy(t). \end{cases}$$

If the eigenvalues are negative, say $\lambda_1 = -\omega_1^2$ and $\lambda_2 = -\omega_2^2$, then the general solution of the second order system is

$$\mathbf{x}(t) = (a_1\cos(\omega_1 t) + b_1\sin(\omega_1 t))\mathbf{v}_1 + (a_2\cos(\omega_2 t) + b_2\sin(\omega_2 t))\mathbf{v}_2.$$

1. First Order Linear System. Consider the following system of differential equations:

$$\begin{cases} 5x'(t) &= x(t) + 6y(t), \\ 5y'(t) &= 4x(t) - y(t). \end{cases}$$

(a) Find the 2×2 matrix A such that $\mathbf{x}'(t) = A\mathbf{x}(t)$, where $\mathbf{x}(t) = (x(t), y(t))$.
(b) Find the eigenvalues and eigenvectors of A.
(c) Use part (b) to find the general solution of the system.
(d) Find the specific solution with initial conditions $x(0) = 0$ and $y(0) = 5$.

(a): We have

$$\begin{aligned} &\begin{cases} 5x'(t) &= x(t) + 6y(t), \\ 5y'(t) &= 4x(t) - y(t), \end{cases} \\ \rightsquigarrow \quad &\begin{cases} x'(t) &= x(t)/5 + 6y(t)/5, \\ 5y'(t) &= 4x(t)/5 - y(t)/5, \end{cases} \end{aligned}$$

$$\rightsquigarrow \quad \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 1/5 & 6/5 \\ 4/5 & -1/5 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$\rightsquigarrow \quad \mathbf{x}'(t) = \begin{pmatrix} 1/5 & 6/5 \\ 4/5 & -1/5 \end{pmatrix} \mathbf{x}(t).$$

(b): The eigenvalues are the solutions of the characteristic equation:

$$\begin{aligned} \begin{vmatrix} 1/5-\lambda & 6/5 \\ 4/5 & -1/5-\lambda \end{vmatrix} &= 0 \\ (1/5-\lambda)(-1/5-\lambda)-(4/5)(6/5) &= 0 \\ \lambda^2 - 1/25 - 24/25 &= 0 \\ \lambda^2 - 1 &= 0 \\ (\lambda-1)(\lambda+1) &= 0 \\ \lambda &= \pm 1. \end{aligned}$$

The eigenvectors for $\lambda = 1$ satisfy

$$\begin{pmatrix} 1/5 & 6/5 \\ 4/5 & -1/5 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 1 \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\rightsquigarrow \quad \begin{cases} (1/5-1)u + (6/5)v &= 0, \\ (4/5)u + (-1/5-1)v &= 0, \end{cases}$$

$$\rightsquigarrow \quad \begin{cases} (-4/5)u + (6/5)v &= 0, \\ (4/5)u + (-6/5)v &= 0, \end{cases}$$

$$\rightsquigarrow \quad \begin{cases} -2u + 3v &= 0, \\ 2u - 3v &= 0. \end{cases}$$

We can choose any solution we want. Let's take $(u, v) = (3, 2)$. We verify that

$$\begin{pmatrix} 1/5 & 6/5 \\ 4/5 & -1/5 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 15/5 \\ 10/5 \end{pmatrix} = 1 \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

The eigenvectors for $\lambda = -1$ satisfy

$$\begin{pmatrix} 1/5 & 6/5 \\ 4/5 & -1/5 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = -1 \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\rightsquigarrow \quad \begin{cases} (1/5+1)u + (6/5)v &= 0, \\ (4/5)u + (-1/5+1)v &= 0, \end{cases}$$

$$\rightsquigarrow \quad \begin{cases} (6/5)u + (6/5)v &= 0, \\ (4/5)u + (4/5)v &= 0, \end{cases}$$

$$\rightsquigarrow \quad \begin{cases} u + v &= 0, \\ u + v &= 0. \end{cases}$$

We can choose any solution we want. Let's take $(u, v) = (1, -1)$. We verify that

$$\begin{pmatrix} 1/5 & 6/5 \\ 4/5 & -1/5 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -5/5 \\ 5/5 \end{pmatrix} = -1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

(c): Based on part (b), the general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} = \begin{pmatrix} 3c_1 e^t + c_2 e^{-t} \\ 2c_1 e^t - c_2 e^{-t} \end{pmatrix}.$$

(d): Substituting the initial conditions $x(0) = 0$ and $y(0) = 5$ gives

$$\begin{aligned} \begin{pmatrix} 0 \\ 5 \end{pmatrix} &= \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 3c_1 + c_2 \\ 2c_1 - c_2 \end{pmatrix} \\ \rightsquigarrow \quad & \left\{ \begin{array}{rcl} 0 &=& 3c_1 + c_2, \\ 5 &=& 2c_1 - c_2, \end{array} \right. \\ \rightsquigarrow \quad & \left\{ \begin{array}{rcl} c_1 &=& 1, \\ c_2 &=& -3. \end{array} \right. \end{aligned}$$

Hence the solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = 1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^t - 3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} = \begin{pmatrix} 3e^t - 3e^{-t} \\ 2e^t + 3e^{-t} \end{pmatrix}.$$

2. Second Order Linear System. Consider the following system of differential equations:

$$\left\{ \begin{array}{rcl} x''(t) &=& -2x(t) + 2y(t), \\ y''(t) &=& x(t) - 3y(t). \end{array} \right.$$

(a) Find the 2×2 matrix A such that $\mathbf{x}''(t) = A\mathbf{x}(t)$, where $\mathbf{x}(t) = (x(t), y(t))$.
(b) Find the eigenvalues and eigenvectors of A.
(c) Use (b) to find the general solution of the system.
(d) Find the specific solution with initial conditions $x(0) = x'(0) = y(0) = 0$ and $y'(0) = 1$.

(a): We have

$$\begin{aligned} & \left\{ \begin{array}{rcl} x''(t) &=& -2x(t) + 2y(t), \\ y''(t) &=& x(t) - 3y(t), \end{array} \right. \\ \rightsquigarrow \quad & \begin{pmatrix} x''(t) \\ y''(t) \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \\ \rightsquigarrow \quad & \mathbf{x}''(t) = \begin{pmatrix} -2 & 2 \\ 2 & -3 \end{pmatrix} \mathbf{x}(t). \end{aligned}$$

(b): The eigenvalues are the solutions of the characteristic equation:

$$\begin{aligned} \begin{vmatrix} -2 - \lambda & 2 \\ 1 & -3 - \lambda \end{vmatrix} &= 0 \\ (-2 - \lambda)(-3 - \lambda) - (1)(2) &= 0 \\ \lambda^2 + 5\lambda + 6 - 2 &= 0 \\ \lambda^2 + 5\lambda + 4 &= 0 \\ (\lambda + 1)(\lambda + 4) &= 0 \\ \lambda &= -1, -4. \end{aligned}$$

The eigenvectors for $\lambda = -1$ satisfy

$$\begin{aligned} \begin{pmatrix} -2 & 2 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} &= -1 \begin{pmatrix} u \\ v \end{pmatrix} \\ \rightsquigarrow \quad & \left\{ \begin{array}{rcl} (-2 + 1)u + (2)v &=& 0, \\ (1)u + (-3 + 1)v &=& 0, \end{array} \right. \end{aligned}$$

$$\rightsquigarrow \begin{cases} -u + 2v &= 0, \\ u - 2v &= 0. \end{cases}$$

We can choose any solution we want. Let's take $(u, v) = (2, 1)$. We verify that

$$\begin{pmatrix} -2 & 2 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} = -1 \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

The eigenvectors for $\lambda = -4$ satisfy

$$\begin{pmatrix} -2 & 2 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = -4 \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\rightsquigarrow \begin{cases} (-2+4)u + (2)v &= 0, \\ (1)u + (-3+4)v &= 0, \end{cases}$$

$$\rightsquigarrow \begin{cases} 2u + 2v &= 0, \\ u + v &= 0. \end{cases}$$

We can choose any solution we want. Let's take $(u, v) = (1, -)$. We verify that

$$\begin{pmatrix} -2 & 2 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \end{pmatrix} = -4 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

(c): The eigenvalues $\lambda = -1, -4$ correspond to frequencies $\omega = \sqrt{-\lambda} = 1, 2$. Then based on part (b) the general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = (a_1 \cos t + b_1 \sin t) \begin{pmatrix} 2 \\ 1 \end{pmatrix} + (a2 \cos(2t) + b_2 \sin(2t)) \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

(d): Substituting initial positions $x(0) = y(0) = 0$ into the solution gives

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = (a_1 + 0) \begin{pmatrix} 2 \\ 1 \end{pmatrix} + (a2 + 0) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\rightsquigarrow \begin{cases} 0 &= 2a_1 + a2, \\ 0 &= a_1 - a_2, \end{cases}$$

$$\rightsquigarrow \begin{cases} a_1 &= 0, \\ a_2 &= 0. \end{cases}$$

Now we compute the derivative of the solution:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = (0 \cos t + b_1 \sin t) \begin{pmatrix} 2 \\ 1 \end{pmatrix} + (0 \cos(2t) + b_2 \sin(2t)) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = b_1 \sin(t) \begin{pmatrix} 2 \\ 1 \end{pmatrix} + b_2 \sin(2t) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = b_1 \cos(t) \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 2b_2 \cos(2t) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Finally, we substitute the initial velocities $x'(0) = 0$ and $y'(0) = 1$:

$$\begin{pmatrix} x'(0) \\ y'(0) \end{pmatrix} = b_1 \cos(0) \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 2b_2 \cos(0) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} = b_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 2b_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\rightsquigarrow \begin{cases} 0 &= 2b_1 + 2b_2, \\ 1 &= b_1 - 2b_2, \end{cases}$$

$$\rightsquigarrow \quad \begin{cases} b_1 &= 1/3, \\ b_2 &= -1/3. \end{cases}$$

Hence the solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{1}{3}\sin(t)\begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{1}{3}\sin(2t)\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} (2/3)\sin t - (1/3)\sin(2t) \\ (1/3)\sin t + (1/3)\sin(2t) \end{pmatrix}.$$

Remark: A system of second order linear ODEs represents a system of coupled oscillators. The eigenvectors of the system correspond to the "normal modes of oscillation". The general solution is a "superposition of normal modes". Here are the normal modes for our problem:

`https://www.math.miami.edu/~armstrong/311sp23/hw6_2_mode1.gif`

`https://www.math.miami.edu/~armstrong/311sp23/hw6_2_mode2.gif`

And here is our solution, which is the superposition of the two modes:

`https://www.math.miami.edu/~armstrong/311sp23/hw6_2_modes_superposition.gif`